

AN ANTICYCLOTOMIC MAZUR-TATE CONJECTURE FOR MODULAR FORMS

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ABSTRACT. Under certain assumptions, we prove an anticyclotomic analogue of the “weak main conjecture” à la Mazur and Tate for modular forms over a large class of cyclic ring class extensions.

1. INTRODUCTION

1.1. Overview. The purpose of this article is to prove a Mazur-Tate type conjecture for modular forms at an ordinary prime over cyclic ring class extensions of an imaginary quadratic field. It is a refinement of the Euler system divisibility of the main conjecture of Iwasawa theory for modular forms over anticyclotomic \mathbb{Z}_p -extensions in [BD05]. Here, the *refinement* means that we look at finite extensions, not the infinite \mathbb{Z}_p -extension.

It is known that the Euler system divisibility of the Iwasawa main conjecture implies the corresponding Mazur-Tate conjecture for p -stabilized newforms for cyclic ring class extensions of p -power degree if we have the exact control theorem and the non-existence of a non-trivial finite submodule over the Iwasawa algebra.

Modelling this strategy, we extend the result on the Mazur-Tate conjecture to *general* cyclic ring class extensions. In order to do this, we first prove the Euler system divisibility of the *generalized* Iwasawa main conjecture allowing prime-to- p cyclic ring class extensions. Then we apply the exact control theorem and the non-existence of a non-trivial finite submodule over the generalized Iwasawa algebra to this generalized Euler system divisibility. Here, the *generalized* means that the corresponding Galois group is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}/m'\mathbb{Z}$ with $(p, m') = 1$.

Also, we compare the Bertolini-Darmon elements arising from newforms and their p -stabilizations to obtain the Mazur-Tate type conjecture result for non- p -stabilized forms. In the process, we also observe a different type of the exceptional zero phenomenon arising from prime-to- p extensions (§8).

The proof of this generalized Euler system divisibility *reveals* the significance of the Euler system argument à la Bertolini-Darmon [BD05, §4] *again*. Throughout more than a decade, the formalism of the Euler system argument is extensively used in various settings. The list includes [BD05, §4], [DI08, §5], [PW11, §4.4], [Lon12, §7.6], [vO13, §7], [Gom14, §2.3], [CH15, §6], and [Wan15, §7.3]. The original Euler system argument consists of two parts. The main part is the “character by character” divisibilities over discrete valuation rings. Here, the discrete valuation rings come from the evaluation of characters on the Iwasawa algebra. The other part is the lifting argument from the divisibilities over discrete valuation rings to the desired divisibility over the Iwasawa algebra. As a by-product of the original Euler system argument, one obtains the Euler system divisibility of the main conjecture, which gives an upper bound of the corresponding Selmer groups.

The key observation for the generalization is that the main body of argument for the “character by character” divisibilities over discrete valuation rings is (almost) independent of the shape of the (infinite) extension (modulo some detail, of course) (§15). This is because the main part of the argument is proceeded over discrete valuation rings after specialization via a character. It is also observed in [Nek12] and [Gom14].

We also observe that the serious use of the theory of Iwasawa modules is only made in the lifting argument ([BD05, Proposition 3.1]) from the “character by character” divisibilities to the bona fide divisibility. We also generalize the lifting argument to allow prime-to- p extensions (Proposition 6.2).

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However, it turns out that the backbone and detail of the Euler system argument depends heavily on the properties of p -extensions. Several details of the Euler system argument are not valid anymore in the general cyclic case. For example, all the arguments using the properties of local rings (e.g. Nakayama's lemma with the Iwasawa algebra) simply do not work since the generalized Iwasawa algebra is only semi-local. In order to overcome this issue, we upgrade the Chebotarev density type result (Theorem 13.1) by allowing more general base fields. Using this upgraded tool, we make the reduction process to the base field in the Euler system argument more "relative".

In the reduction process, another key ingredient is the control theorem of compact Selmer groups with respect to quotients (Corollary 14.15), which is based on the freeness theorem of compact Selmer groups (Theorem 14.8). The original proof of the freeness theorem also depends on the properties of p -extensions. In order to generalize the freeness theorem, we generalize the concept of n -admissible sets (Definition 13.2) allowing more general base fields as before. Also, we use the enhanced isotypic decomposition with respect to quotients (§4.3) to recycle the original approach.

Using these ideas, we carefully examine the generalized Euler system argument over discrete valuations rings and apply the generalized lifting argument. To sum up, we do not only use the Euler system argument but also refine the argument itself. We also remark that there is a close similarity between the enhanced isotypic decomposition and Bertolini-Darmon's Euler system argument because both use the specialization via characters.

As a by-product of the generalization, we deduce the Euler system divisibility of the generalized main conjecture and a Mazur-Tate type conjecture for modular forms under cyclic ring class extensions. As a consequence, we obtain more refined and general arithmetic applications than those of the main conjecture. It seems that all the proven Euler system divisibilities mentioned in the above list can be upgraded as ours without introducing new difficulty, at least to some extent. Also, we can obtain the upper bound of M -Selmer groups of elliptic curves over cyclic ring class extensions for some M which is not necessarily a prime power under a certain normalization assumption (§17).

We also give a detailed exposition of the mod p^n level raising at two n -admissible primes (§10.3) and reveals the importance of "Condition Ihara" (Assumption 2.8), which is not emphasized in the former literatures, in the non-ordinary setting (§2.2).

This work can be regarded as a part of the refined Iwasawa theory à la Kurihara [Kur03, §10, §11], [Kur14, Remark 1, §1] for the anticyclotomic setting. Note that our approach is completely different from the current approaches to the cyclotomic Mazur-Tate conjecture (for examples, [Ota17], [EPW]) mainly due to the significant difference between Euler systems arising from Heegner points and Kato's Euler systems arising from Siegel units.

In the non-ordinary setting, we observed a variety of obstacles including the lack of the lifting argument for cyclic group rings (not Iwasawa algebra), the breakdown of the freeness of Selmer groups (due to the lack of control theorem) and no existence of a good local Iwasawa theory for the p -inert case. See [Ber14] for the obstacle in the p -inert case. It is now known that the Bloch-Kato local condition is well-behaved under congruences at least for the low weight case. See [Nek12, Appendix A], [IM15], and [Chi16, Proposition 7.3] for the congruence behavior of the Bloch-Kato Selmer groups.

Remark 1.1 (on the generality and the convention). As we already mentioned, there are various settings where the Euler system argument is used. We adapt and refine the setting of [PW11], namely ordinary modular forms of weight 2 with arbitrary Fourier coefficients. However, we adapt the language of [CH15] mostly.

1.2. Setting the stage.

1.2.1. Modular setup. Let $p > 3$ be a prime and N be an integer ≥ 1 with $(N, p) = 1$. Fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $f = f(\tau) \in S_2(\Gamma_0(N))^{\text{new}}$ be a p -ordinary newform. Let E be the Hecke field of f over \mathbb{Q}_p , which is compatible with ι_p , and \mathcal{O} the ring of integers of E . Let ϖ be a uniformizer of \mathcal{O} , $\mathbb{F} := \mathcal{O}/\varpi\mathcal{O}$ the residue field, and $\mathcal{O}_n := \mathcal{O}/\varpi^n$. For a field F , denote the absolute Galois group of F by G_F . Let

$$\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(E) = \text{GL}(V_f)$$

be the Galois representation associated to f . Let $T_f \subset V_f$ be a Galois-stable \mathcal{O} -lattice of f . We assume that the residual representation

$$\overline{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}) = \text{GL}(T_f/\varpi T_f)$$

of f is absolutely irreducible. Then $\bar{\rho}$ is independent of the choice of T_f . Let $T_{f,n} := T_f/\varpi^n T_f$, $A_f := V_f/T_f$, and $A_{f,n} := A_f[\varpi^n]$. It is known that $A_{f,n} \simeq T_{f,n}$ and $A_{f,n} \simeq \text{Hom}_{\mathcal{O}}(T_{f,n}, E/\mathcal{O}(1))$ as $G_{\mathbb{Q}}$ -modules via the self-duality. By the ordinary assumption, we define the p -stabilized newform

$$f_{\alpha}(\tau) := f(\tau) - \frac{1}{\alpha} f(p\tau)$$

of f with the unit U_p -eigenvalue $\alpha = \alpha_p(f)$.

1.2.2. Anticyclotomic setup. Fix an imaginary quadratic field K with $(\text{disc}(K), Np) = 1$. Let $N(\bar{\rho})$ be the prime-to- p conductor of $\bar{\rho}$. Then we have decompositions

$$\begin{aligned} N &= N^+ N^- \\ N(\bar{\rho}) &= N(\bar{\rho})^+ N(\bar{\rho})^- \end{aligned}$$

where the $(+)$ -part is the product of the primes which splits in K and the $(-)$ -part is the product of the primes which is inert in K . Note that $N(\bar{\rho})^{\pm}$ divides N^{\pm} , respectively.

Assumption 1.2 (Parity). In the decomposition, N^- is the square-free product of an odd number of primes.

Remark 1.3. The parity assumption implies that we deal with the opposite case to [Dar92, Conjecture 2.1 and Theorem 2.4]. See also [LV17, Theorem 1.1 and Conjecture 5.1] for the higher weight case.

Let $m = m'p^r$ be a non-negative integer with $(m', p) = 1$ such that $(m, N) = 1$. For $m = \prod \ell_i^{r_i}$, write $m_0 = \prod \ell_i^{r_i+1}$ where ℓ runs over primes. Let $H(m_0)$ be the ring class field of K of conductor m_0 , $G_{m_0} := \text{Gal}(H(m_0)/K)$. For any prime ℓ , let $K(\ell^r) \subset H(\ell^{r+1})$ be the maximal ℓ -subextension of $H(\ell^{r+1})$ over K . Let $K(m)$ be the compositum of the $K(\ell_i^{r_i})$'s. Then $K(m)$ is a cyclic extension of K and is contained in $H(m_0)$.

Remark 1.4. If ℓ is totally ramified in $K(\ell^r)/K$, then the extension degree of $K(\ell^r)$ over K is ℓ^r . The index $[H(\ell^{r+1}) : K(\ell^r)]$ is $\ell + 1/\ell - 1$ if ℓ is inert/splits in K/\mathbb{Q} , respectively.

Remark 1.5. From now on, we assume that every prime ℓ_i dividing m are totally ramified in $K(\ell_i^{r_i})/K$ just for the notational convenience (i.e. for counting the exact degree).

Let $\Gamma_m := \text{Gal}(K(m)/K)$ and then Γ_m is a cyclic group of order m . We fix a generator $\gamma_m \in \Gamma_m$ and isomorphism $\mathcal{O}[\Gamma_m] \simeq \mathcal{O}[T]/((1+T)^m - 1)$ defined by $\gamma_m \mapsto 1+T$.

Let $K(p^{\infty}) := \bigcup_{r=1}^{\infty} K(p^r)$ be the anticyclotomic \mathbb{Z}_p -extension and $\Gamma_{p^{\infty}} := \text{Gal}(K(p^{\infty})/K)$. Let $\Lambda_{p^{\infty}} = \mathcal{O}[[\Gamma_{p^{\infty}}]]$ be the Iwasawa algebra.

Similarly, let $K(m'p^{\infty}) := \bigcup_{r=1}^{\infty} K(m'p^r)$ and $\Lambda_{m'p^{\infty}} := \mathcal{O}[[\Gamma_{m'p^{\infty}}]] \simeq \mathcal{O}[\Gamma_{m'}][[\Gamma_{p^{\infty}}]]$ be the generalized Iwasawa algebra.

1.3. Anticyclotomic Mazur-Tate conjectures for modular forms. We state an anticyclotomic analogue of the Mazur-Tate conjecture [MT87], which we prove under certain conditions. Let $L_p(K(m), f) \in \mathcal{O}[\text{Gal}(K(m)/K)]$ be the Bertolini-Darmon element attached to f and $K(m)$ defined in §3, which is an analogue of Mazur-Tate elements. Observing the statement of the Iwasawa main conjecture, we can easily state the following conjecture.

Conjecture 1.6 (“Weak main conjecture” à la Mazur-Tate: the minimal Selmer case).

$$L_p(K(m), f) \in \text{Fitt}_{\mathcal{O}[\text{Gal}(K(m)/K)]}(\text{Sel}(K(m), A_f)^{\vee})$$

where $\text{Sel}(K(m), A_f)$ is the minimal Selmer group as in §5.5.

It seems that there is no direct way to attack this minimal Selmer case. The practical Euler system argument in all the literatures works only well with the N^- -ordinary Selmer group $\text{Sel}_{N^-}(K(m), A_{f,n})$ (Definition 5.2). Thus, we consider the following formulation.

Conjecture 1.7 (“Weak main conjecture” à la Mazur-Tate: the N^- -ordinary Selmer case).

$$L_p(K(m), f) \in \text{Fitt}_{\mathcal{O}[\text{Gal}(K(m)/K)]}(\text{Sel}_{N^-}(K(m), A_f)^{\vee})$$

where $\text{Sel}_{N^-}(K(m), A_f) := \varprojlim_n \text{Sel}_{N^-}(K(m), A_{f,n})$ and the transition map with respect to n is induced from the embedding $A_{f,n} \hookrightarrow A_{f,n+1}$.

Remark 1.8 (on the coefficients). In the very original formulation of the Mazur-Tate conjecture, the coefficient ring is a subring of a global field. In our formulation, we assume that the coefficient ring \mathcal{O} is p -adic. See §17 for this global aspect.

1.4. Running hypotheses. We make the list of assumptions we need for our main theorem and point out where the conditions are used throughout this article.

Assumption 1.9 (N^\pm -minimality). The form $f \in S_2(\Gamma_0(N))^{\text{new}}$ is N^\pm -**minimal** if $N^\pm = N(\bar{\rho})^\pm$, respectively.

Remark 1.10. The N^+ -minimality assumption is essentially used in control theorems (Lemma 14.2 and Lemma 14.4), and these lemma play important roles in the proof of Theorem 14.8. Without the N^+ -minimality assumption, Theorem 14.8 may fail. See [KPW17] for this issue. Also, note that the N^+ -minimality assumption corresponds to [BD94, Assumption 2.15] for primes dividing N^+ .

Condition CR is a relaxation of N^- -minimality but still ensures a mod p multiplicity one result for the localized Hecke modules we deal with.

Assumption 1.11 (Condition CR: [Hel07, Definition 6.4], [Kim17, Definition 1.6]). Let $\bar{\tau}$ be a residual modular Galois representation with residual characteristic ≥ 5 . Let M be a square-free product of an odd number of primes and is divisible by $N(\bar{\tau})^-$. The pair $(\bar{\tau}, M)$ satisfies **condition CR** if

- $\bar{\tau}$ is irreducible, and
- if $q \mid M$ and $q \equiv \pm 1 \pmod{p}$, then $\bar{\rho}$ is ramified at q , i.e. $q \mid N(\bar{\tau})^-$.

Remark 1.12. Condition CR is broadly used in §10 to obtain the uniqueness of mod p^n modular forms under level raising.

We impose the following certain local condition at p , which is slightly stronger than Hypothesis (PO) in [CH15].

Let $\alpha_p(f)$ be the unit root of the Hecke polynomial $X^2 - a_p(f)X + p$ of f at p . If p splits in K , we decompose $p = \mathfrak{p}\bar{\mathfrak{p}}$ in K . Let $\text{Fr}_{\mathfrak{p}}$ and $\text{Fr}_{\bar{\mathfrak{p}}}$ be the Frobenii in $\Gamma_{m'}$ under the global reciprocity map with geometric normalization.

Assumption 1.13 (Condition PO). The triple $(f, K(m')/K, p)$ satisfies **condition PO** if

- (1) p is inert in K and

$$a_p(f) \not\equiv \pm 1 \pmod{\varpi},$$

or

- (2) p splits in K and

$$\omega(\text{Fr}_{\mathfrak{p}}) \not\equiv \alpha_p(f) \pmod{\varpi_\omega}$$

and

$$\omega(\text{Fr}_{\bar{\mathfrak{p}}}) \not\equiv \alpha_p(f) \pmod{\varpi_\omega}$$

for any character $\omega : \Gamma_{m'} \rightarrow \overline{\mathbb{Q}}_p^\times$ where ϖ_ω is a uniformizer of the extension of \mathcal{O} adjoining the values of ω .

Remark 1.14. (1) It corresponds to [BD94, Assumption 2.15] for primes dividing p via [BD94, Proposition 2.16]. Thus, it is used in the exact control theorem in the algebraic side. See Lemma 5.12 and Lemma 14.2.

(2) It removes the “tame exceptional zero phenomenon” in the analytic side. See §8.

(3) It is slightly weaker than Hypothesis (PO) in [CH15] if $m' = 1$.

Assumption 1.15 (Big image). The residual representation $\bar{\rho}$ of f **has big image** if the image of $\bar{\rho}$ contains a conjugate of $\text{GL}_2(\mathbb{F}_p)$ or $p > 5$.

Remark 1.16. The big image assumption is required for Theorem 13.1. The absolute irreducibility of $\bar{\rho}$, $p > 5$, and Condition CR imply that the image is large enough to establish Theorem 13.1 without presumably knowing $\text{GL}_2(\mathbb{F}_p) \subset \bar{\rho}(G_{\mathbb{Q}})$ up to conjugation. See [CH15, Lemma 6.1 and Lemma 6.2] for detail.

1.5. Main theorems and arithmetic consequences. We state the main theorems and their arithmetic consequences.

Theorem 1.17 (Main theorem I: generalized Iwasawa main conjecture). *Assume the following conditions:*

- (1) $(\bar{\rho}, N^-)$ satisfies Condition CR;
- (2) $(f, K(m')/K, p)$ satisfies Condition PO;
- (3) $\bar{\rho}$ has big image;
- (4) f is N^+ -minimal;
- (5) m' is prime to Np ;

Then

$$L_p(K(m'p^\infty), f_\alpha) \in \text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}(K(m'p^\infty), A_f)^\vee)$$

holds.

Theorem 1.18 (Main theorem II: Mazur-Tate conjecture). *Assume the following conditions:*

- (1) $(\bar{\rho}, N^-)$ satisfies Condition CR;
- (2) $(f, K(m')/K, p)$ satisfies Condition PO;
- (3) $\bar{\rho}$ has big image;
- (4) f is N^+ -minimal;
- (5) m is prime to N .

Then Conjecture 1.7

$$L_p(K(m), f) \in \text{Fitt}_{\mathcal{O}[\text{Gal}(K(m)/K)]}(\text{Sel}_{N^-}(K(m), A_f)^\vee)$$

holds. If we further assume that f is also N^- -minimal (so N -minimal), then Conjecture 1.6

$$L_p(K(m), f) \in \text{Fitt}_{\mathcal{O}[\text{Gal}(K(m)/K)]}(\text{Sel}(K(m), A_f)^\vee)$$

holds.

Remark 1.19. See [BD05, Corollary 2, 3, 4, and 5, Introduction] for the arithmetic applications of the Euler system divisibility. See §2 for the current status of the main conjecture.

Before stating refined arithmetic applications of the main theorems, we briefly review the relevant background on group ring $\mathcal{O}[\Gamma_m]$. See [MT87, (1.5)] for detail.

Let $\chi : \text{Gal}(K(m)/K) \rightarrow \mathcal{O}_\chi^\times$ be a ring class character of finite order where \mathcal{O}_χ is the \mathcal{O} -algebra generated by the values of χ . We regard χ as the \mathcal{O} -algebra homomorphism $\chi : \mathcal{O}[\Gamma_m] \rightarrow \mathcal{O}_\chi$ defined by $\chi : \gamma_m \mapsto \chi(\gamma_m)$. Let $I_\chi := \ker(\chi) \subseteq \mathcal{O}[\Gamma_m]$ be the augmentation ideal at χ .

Definition 1.20 (Vanishing order). Let $\theta \in \mathcal{O}[\Gamma_m]$. We say θ **vanishes to infinite order at χ** if θ is contained in any power of I_χ . We say θ **vanishes to order r at χ** if $\theta \in I_\chi^r \setminus I_\chi^{r+1}$.

From now on, assume the Hecke field of f is \mathbb{Q} . Let E_f be an elliptic curve over \mathbb{Q} corresponding to f via [BCDT01, Theorem A]. We also assume the N -minimality of f to have the coincidence of the minimal Selmer group of f , the N^- -ordinary Selmer group of f , and the classical Selmer group of E_f .

The following corollary of Theorem 1.18 follows.

Corollary 1.21 (“Weak vanishing conjecture” à la Mazur-Tate; [MT87, Conjecture 1]). *Under the same conditions of Theorem 1.18 and the N^- -minimality of f , the rank*

$$\text{rank}_{\mathbb{Z}_{p,\chi}}(E_f(K(m))_\chi)$$

is equal to or less than the vanishing order of $L_p(K(m), f)$ at χ where

$$E_f(K(m))_\chi := E_f(K(m)) \otimes_{\mathbb{Z}[\Gamma_m], \chi} \mathbb{Z}_{p,\chi}$$

is the χ -isotypic quotient of Mordell-Weil group $E_f(K(m))$.

Remark 1.22. See [MT87, Proposition 3] for proof. It is a refinement of [BD05, Corollary 2, Introduction].

In the case of vanishing order zero, we obtain a more refined formula from Theorem 1.18.

Corollary 1.23 (Vanishing order zero case). *Assume the same conditions of Theorem 1.18 and the N^- -minimality of f . If $L_p(K(m), f)$ has vanishing order zero at χ , then*

$$\text{ord}_p(\#|E_f(K(m))_\chi|) + \text{ord}_p(\#\text{III}(K(m), E_f)[p^\infty]_\chi) \leq \text{ord}_p(\chi(L_p(K(m), f)))$$

where $\text{III}(K(m), E_f)[p^\infty]_\chi$ is the χ -isotypic quotient of the p -part of the Shafarevich-Tate group of E_f over $K(m)$.

Remark 1.24. It is the weak vanishing conjecture at χ with vanishing order zero and also a refinement of [BD05, Corollary 4, Introduction]. This corollary gives a partial answer to the question raised in [BD05, Remark 1 to Corollary 4, Introduction]. See §17 for another aspect of the question. Note that the proof of this corollary itself does not appeal to any interpolation formula à la Gross-Walspurger-Zhang at all. However, the nonvanishing of $L_p(K(m), f)$ at almost all χ requires [Hun17, Theorem C and §6.1] generalizing [Vat02, Theorem 1.4].

We state a bit more *global* application of our main theorem. See §17 for detail.

Let $M = \prod_i p_i^{r_i}$ be an integer where $p_i > 3$ is a prime such that

- (1) each p_i is good ordinary for E ;
- (2) each residual representation $\bar{\rho}_{E, p_i}$ is surjective;
- (3) condition PO holds for $(E, K(m'_i)/K, p_i)$ for each p_i where m'_i is the prime-to- p_i part of m ;
- (4) $N = N(\bar{\rho}_{E, p_i})$ for each p_i .

Let f be the newform corresponding to E and assume that the Jacquet-Langlands correspondence ensures the integral normalization of the corresponding quaternionic form for all p_i . See Assumption 17.1 for detail. Let $L_M(K(m), E)$ be the partially adelic anticyclotomic L -function defined in §17 under Assumption 17.1.

Theorem 1.25 (Theorem 1.18 for M -Selmer groups). *Under Assumption 17.1, we have*

$$L_M(K(m), E) \in \text{Fitt}_{\mathbb{Z}/M\mathbb{Z}[\Gamma_m]}(\text{Sel}(K(m), E[M])^\vee)$$

in $(\mathbb{Z}/M\mathbb{Z})[\Gamma_m]$ where $(-)^\vee = \text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ here.

In the same manner as in Corollary 1.23, we obtain

Corollary 1.26 (Corollary 17.4; Corollary 1.23 for M -Selmer groups). *Suppose that Assumption 17.1 holds. Let $\chi : \Gamma_m \rightarrow \overline{\mathbb{Q}}^\times$ be a character and assume that $\chi(L_M(K(m), f)) \neq 0$. Then*

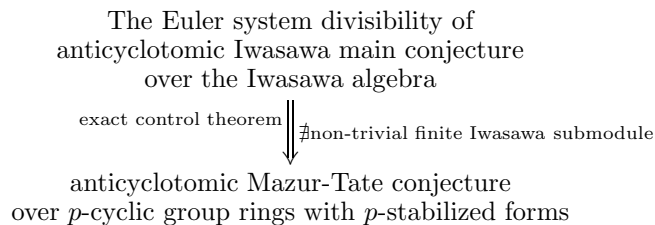
$$\sum_{p_i} (\text{ord}_{p_i}(\#|E(K(m))_\chi|) + \text{ord}_{p_i}(\#\text{III}(K(m), E)[M]_\chi)) \leq \sum_{p_i} \text{ord}_{p_i}(\chi(L_M(K(m), f)))$$

where $\text{III}(K(m), E)[M]_\chi$ is the χ -isotypic quotient of the M -torsion subgroup of the Shafarevich-Tate group of E over $K(m)$.

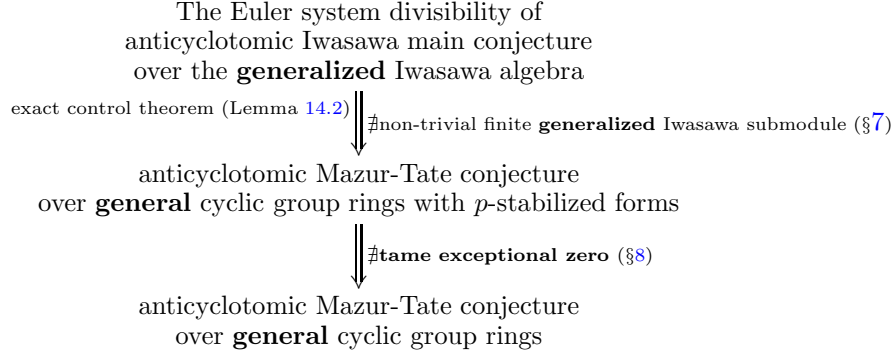
1.6. The strategy of proof and the structure of this article.

1.6.1. *The strategy.* We follow the strategy of [BD05] and the notation of [CH15] very closely, but we will explicitly point out the differences when we encounter them.

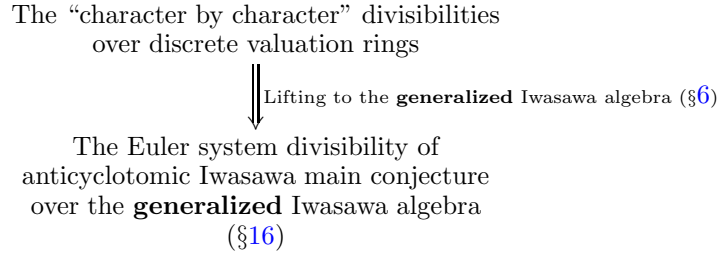
The following diagram shows the strategy of proof and the detail is given in Proposition 7.1.



Modelling the above strategy, we extend the result to general cyclic ring class extensions as follows:

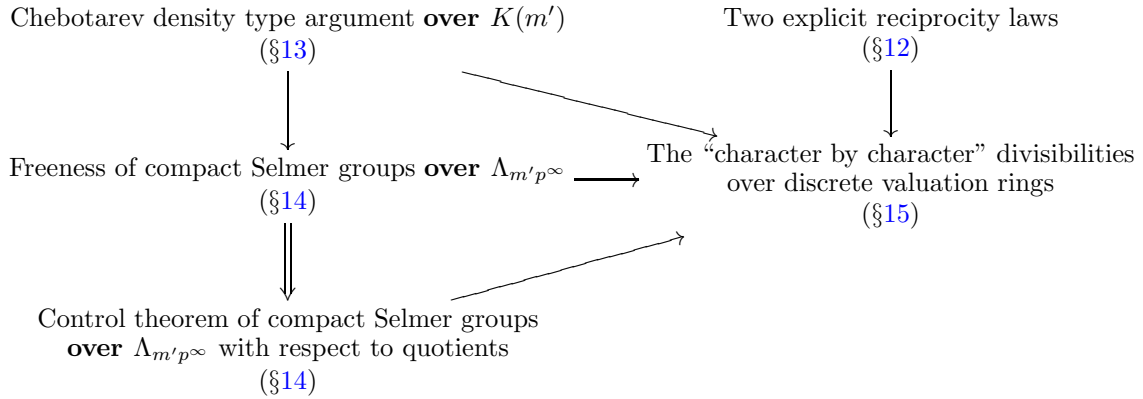


The reduction of the Euler system divisibility to the Mazur-Tate conjecture for the p -stabilized forms is explained in §7 and its reduction to the conjecture for the non- p -stabilized forms is given in §8. Here, the *generalized* Iwasawa algebra means that we allow finite prime-to- p extensions in the infinite Galois extension. In other words, it is isomorphic to $\mathbb{Z}_p[\Gamma_{m'}][[T]]$ where $\Gamma_{m'}$ is the cyclic group of order m' with $(m', p) = 1$.

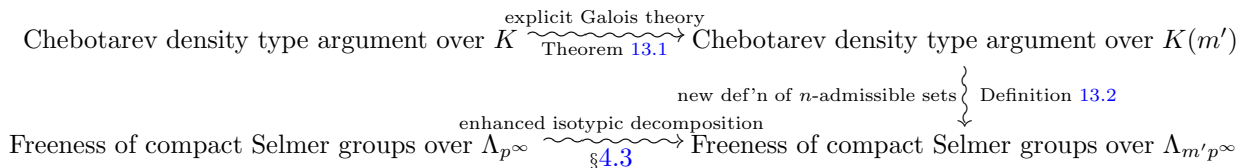


The generalized Euler system divisibility comes from the character by character divisibilities and the lifting argument, which is given in §6.

In order to obtain the character by character divisibilities, the following diagram shows the strategy.



In the above diagram, the Chebotarev density type argument over $K(m')$ and the freeness of compact Selmer groups over $\Lambda_{m'p^\infty}$ require some work for the generalization. The $m' = 1$ case is only covered in the former literature. The following diagram points out how the generalizations are made.



1.6.2. *The structure.* We summarize the content of each section.

In §2, we briefly review the current state of art of the main conjecture.

In §3, we quickly review modular forms of weight two in the quaternionic setting and the construction of the Bertolini-Darmon elements, which is the anticyclotomic analogue of Mazur-Tate elements.

In §4, we quickly review the Fitting ideals, fix the convention on various characters, and also review the enhanced isotypic decomposition, which plays an important role in §14.3. This is one of the key ingredients.

In §5, we recall Δ -ordinary S -relaxed Selmer groups and collect relevant facts on local and global Galois cohomology. The concept of n -admissible primes is introduced.

In §6, we discuss the generalized divisibility criterion, which lifts the character by character result to the main result. This generalization is another ingredient for the generalization. The non-existence of nontrivial finite $\mathcal{O}[\Gamma_{m'p^\infty}]$ -submodules plays an important role here.

In §7, we explain how to deduce the Mazur-Tate conjecture for p -stabilized forms from the generalized Euler system divisibility.

In §8, we explain how to deduce the Mazur-Tate conjecture for non- p -stabilized forms from the Mazur-Tate conjecture for p -stabilized forms under Assumption 1.13.

In §9, we recall the basic notion of Shimura curves and CM points.

In §10, we study two kinds of mod ϖ^n level raising results: level raisings at one n -admissible prime and at two n -admissible primes. We give a detailed and explicit argument for the latter one, which requires some work of Ihara. It turns out that Condition Ihara (Assumption 2.8) is required for the argument in the non-ordinary setting.

In §11, we construct the cohomology classes arising from Heegner points and mod ϖ^n congruences of Hecke eigensystems. We also state the local properties of the cohomology classes.

In §12, we introduce a mod ϖ^n version of theta elements attached to mod ϖ^n modular forms. Then we describe the first and second explicit reciprocity laws.

In §13, we generalize the existence of infinitely many n -admissible primes allowing more general base fields, namely $K(m')$, via the Chebotarev density type argument and also generalize the concept of n -admissible sets over $K(m')$. This is also one of the key ingredients.

In §14, we discuss control theorems and give a detailed proof of the freeness of Δ -ordinary S -relaxed compact Selmer groups over $K(m)$ following [BD94]. For the generalization to $K(m)$, we use the enhanced isotypic decomposition introduced in §4.3.

In §15, we proceed the Euler system argument (via induction). We examine the Euler system argument over $K(m)$. Two parts in the original proof depends heavily on the properties of p -extensions (§15.4 and §15.5). We overcome these issues by using the generalization of the infinitude of n -admissible primes over $K(m')$ in §13. Here, we obtain the character by character result.

In §16, we complete the proof of the main theorem with help of §4.

In §17, we investigate a more global aspect of the Mazur-Tate type conjecture. In fact, it is more coincident with the spirit of the original Mazur-Tate conjecture. See [MT87, Introduction].

2. REMARKS ON THE MAIN CONJECTURE

We briefly review the state of art of the anticyclotomic main conjecture for modular forms of weight 2 up to now and explain how the main conjecture and the exact control theorem imply the Mazur-Tate conjecture for anticyclotomic extensions of p -power conductor.

2.1. Ordinary case. The Euler system method yields the following statement.

Theorem 2.1 ([BD05], [PW11], and [CH15]). *Assume the following conditions:*

- (1) $(\bar{\rho}, N^-)$ satisfies Condition CR.
- (2) $a_p(f) \not\equiv \pm 1 \pmod{\varpi}$. (Condition PO)
- (3) f is N^+ -minimal.
- (4) $\bar{\rho}$ has big image.

Then $L_p(K(p^\infty), f_\alpha) \in \text{char}_\Lambda(\text{Sel}(K(p^\infty), A_f)^\vee)$.

Remark 2.2 (on the N^+ -minimality). In the original proof, the freeness of compact Δ -ordinary S -relaxed Selmer groups over $K(p^\infty)$ is obtained from the freeness of compact Δ -ordinary S -relaxed Selmer groups over $K(p^r)$ by taking the inverse limit. Here, we need the N^+ -minimality assumption.

Remark 2.3 (Condition CR vs. Condition CR⁺). Condition CR⁺, which is more restrictive than condition CR, is imposed in [CH15], but it can be replaced by condition CR for the case of weight 2.

The reason why we need such conditions is to have a mod p multiplicity one result of the localized Hecke module we deal with. Condition CR comes from the arithmetic of Jacobians of Shimura curves and the Jacquet-Langlands correspondence, and Condition CR⁺ comes from the patching argument à la Taylor-Wiles, Diamond, and Fujiwara. One advantage of Condition CR⁺ makes it possible to deal with the higher weight case, but a certain ramification conditions at primes dividing N^+ is imposed.

Using the vanishing of algebraic μ -invariant of the N^+ -minimal case via Theorem 2.1 and the Iwasawa-theoretic argument in [KPW17], one can obtain the following statement.

Proposition 2.4 (Theorem 14.8 over $K(p^\infty)$, [KPW17]). *The freeness of compact Δ -ordinary S -relaxed Selmer groups over $K(p^\infty)$ holds without assuming the N^+ -minimality.*

Recycling the original argument with Proposition 2.4, we can remove N^+ -minimality in Theorem 2.1.

Corollary 2.5. *Assume the following conditions:*

- (1) $(\bar{\rho}, N^-)$ satisfies Condition CR.
- (2) f satisfies Condition PO.
- (3) $\bar{\rho}$ has big image.

Then $L_p(K(p^\infty), f_\alpha) \in \text{char}_\Lambda(\text{Sel}(K(p^\infty), A_f)^\vee)$.

On the other side, the Eisenstein congruence method à la Ribet-Skinner-Urban yields the opposite divisibility.

Theorem 2.6 ([SU14, Theorem 3.37]). *Assume the following conditions:*

- (1) p splits in K .
- (2) $\bar{\rho}$ is N^- -minimal.
- (3) the image of $\bar{\rho}$ contains a conjugate of $\text{GL}_2(\mathbb{F}_p)$.

Then $\text{char}_\Lambda(\text{Sel}(K(p^\infty), A_f)^\vee) \subseteq (L_p(K(p^\infty), f_\alpha))$.

Combining the above two statements, we have the following statement.

Corollary 2.7 (The main conjecture). *Assume the following conditions:*

- (1) p splits in K .
- (2) $\bar{\rho}$ is N^- -minimal.
- (3) f satisfies Condition PO.
- (4) the image of $\bar{\rho}$ contains a conjugate of $\text{GL}_2(\mathbb{F}_p)$.

Then $\text{char}_\Lambda(\text{Sel}(K(p^\infty), A_f)^\vee) = (L_p(K(p^\infty), f_\alpha))$.

2.2. Non-ordinary case. For this subsection, assume that the modular form f is non-ordinary at p . One can obtain a similar result using the \pm -theory à la Kobayashi-Pollack ([Kob03], [Pol03]) under the assumption $a_p(f) = 0$. We need the following condition for the non-ordinary case.

Assumption 2.8 (Condition Ihara). We say that N^+ satisfies **Condition Ihara** if there exists an integer q dividing N^+ such that $q \geq 4$ and p does not divide its Euler totient $\varphi(q)$.

Theorem 2.9 ([DI08], [PW11], and [KPW17]). *Assume the following conditions:*

- (1) p splits in K .
- (2) $(\bar{\rho}, N^-)$ satisfies Condition CR.
- (3) $a_p(f) = 0$.
- (4) $\bar{\rho}$ has big image.
- (5) $K(p^\infty)/K$ is totally ramified.
- (6) N^+ satisfies Condition Ihara.

Then $L_p^\pm(K(p^\infty), f) \in \text{char}_\Lambda(\text{Sel}^\pm(K(p^\infty), A_f)^\vee)$, respectively.

Remark 2.10. Condition (1) and (5) are required to use the local Iwasawa theory for Lubin-Tate extensions following [IP06].

Remark 2.11. Note that Condition Ihara is missing in the former literatures. In order to obtain the second explicit reciprocity law in this setting, Condition Ihara is required since p is not in the level for the non-ordinary setting. Condition Ihara is used for mod ϖ^n level raising at two n -admissible primes

(§10.3). This condition with the $\Gamma_1(q)$ -level structure with $q \geq 4$ makes the arithmetic subgroup defining Shimura curves torsion-free. In some sense, this condition removes the analogue of elliptic points for Shimura curves. The exactly same phenomenon is also observed in the definite case as in [Buz07, §4]. Note that p is in the level in all the former literatures except [DI08] and [PW11, §2.4 and the latter part of Theorem 4.1]. On the other hand, all the former literatures also need to remove p in the level of Hecke modules localized at the maximal ideal using Condition PO to invoke Ihara's lemma. See [CH15, Lemma 5.3] for detail. If N^+ is divisible by 4, then Condition Ihara always holds.

3. MODULAR FORMS AND THE BERTOLINI-DARMON ELEMENTS

We review modular forms of weight two in the quaternionic setting and the theta elements and construct the Bertolini-Darmon elements, which is an anticyclotomic analogue of Mazur-Tate elements. See [Kim17, §2.3] for a more detailed construction.

3.1. Modular forms of weight two. Let $f \in S_2(\Gamma_0(N))^{\text{new}}$ be a normalized newform. Let B be the definite quaternion algebra over \mathbb{Q} of discriminant N^- and $R \subseteq B$ be an oriented Eichler order of level N^+ . For any \mathbb{Z} -module M , $\widehat{M} := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Using an integral version of Jacquet-Langlands correspondence [Kim17, Theorem 2.1 and Remark 2.2], we regard f as a normalized \mathcal{O} -valued modular form on $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$, which is defined up to multiplication by \mathcal{O}^\times . We define the space of \mathcal{O} -valued (quaternionic) modular forms by

$$M_2^{N^-}(N^+) := M_2^{N^-}(N^+, \mathcal{O}) = \{f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \rightarrow \mathcal{O}\}$$

and the space of \mathcal{O} -valued (quaternionic) cuspforms by

$$S_2^{N^-}(N^+) := S_2^{N^-}(N^+, \mathcal{O}) = \{f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \rightarrow \mathcal{O}\} \text{ modulo the constant function}$$

and let $\mathbb{T}^{N^-}(N^+)$ be the Hecke algebra faithfully acting on $S_2^{N^-}(N^+)$. The integral Jacquet-Langlands correspondence identifies

- $\mathbb{T}^{N^-}(N^+)$ and the N^- -new quotient of the classical Hecke algebra $\mathbb{T}(\Gamma_0(N))$, and
- $S_2^{N^-}(N^+)$ and $S_2(\Gamma_0(N))^{N^--\text{new}}$ as Hecke modules.

For each $b \in \widehat{B}^\times$, we write $[b]_U \in B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ represented by b where $U = \widehat{R}^\times$. We have a canonical Hecke-equivariant identification

$$M_2^{N^-}(N^+, \mathbb{Z}) \simeq \mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]$$

where $\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]$ is the divisor group which is a Hecke module via Picard functoriality. Let $\tau \in \widehat{B}^\times$ be the Atkin-Lehner involution defined by $\tau_q = \begin{pmatrix} 0 & 1 \\ N^+ & 0 \end{pmatrix}$ if $q \mid N^+$ and $\tau_q = 1$ otherwise. Then we can explicitly define the canonical identification by $[b]_U \mapsto f_b$ where

$$f_b(b') := \begin{cases} \# \left| (B^\times \cap b \widehat{R}^\times b^{-1}) / \mathbb{Q}^\times \right| & \text{if } [b]_U = [b']_U \\ 0 & \text{if } [b]_U \neq [b']_U \end{cases}$$

We define the perfect pairing

$$\langle -, - \rangle_U : M_2^{N^-}(N^+, A) \times M_2^{N^-}(N^+, A) \rightarrow A$$

where A is any p -adic ring by

$$\langle f_1, f_2 \rangle_U = \sum_{[b]_U} f_1(b) \cdot f_2(b\tau) \cdot \# \left| (B^\times \cap b \widehat{R}^\times b^{-1}) / \mathbb{Q}^\times \right|^{-1}.$$

Remark 3.1. If $p > 3$, $\# \left| (B^\times \cap b \widehat{R}^\times b^{-1}) / \mathbb{Q}^\times \right|$ is always invertible in A . If the level structure of \widehat{R}^\times is smaller than the $\Gamma_1(q)$ -level structure with $q \geq 4$, then $\# \left| (B^\times \cap b \widehat{R}^\times b^{-1}) / \mathbb{Q}^\times \right| = 1$.

Then the action of $\mathbb{T}^{N^-}(N^+)$ on $M_2^{N^-}(N^+)$ is self-adjoint with respect to the pairing $\langle -, - \rangle_U$; we have $\langle t f_1, f_2 \rangle_U = \langle f_1, t f_2 \rangle_U$ for all $t \in \mathbb{T}^{N^-}(N^+)$. By the explicit correspondence, we can also define the pairing

$$\langle -, - \rangle_U : M_2^{N^-}(N^+, A) \times A[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times] \rightarrow A$$

with the same notation and compare with the evaluation as follows:

$$\langle f, b\tau \rangle_U = \langle f, f_{b\tau} \rangle_U = f(b).$$

3.2. Construction of the Bertolini-Darmon elements. Let $\text{rec}_K : K^\times \backslash \widehat{K}^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$ be the reciprocity map in class field theory with geometric normalization. Let $\mathcal{O}_{K,m_0} = \mathbb{Z} + m_0 \mathcal{O}_K$ be the order of K of conductor m_0 for $m_0 \geq 1$. Fix an oriented optimal embedding $\psi : K \rightarrow B$. Then ψ induces a family of maps

$$\psi_{m_0} : K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_{K,m_0}^\times \widehat{\mathbb{Q}}^\times \rightarrow B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$$

for $m_0 \geq 1$ as in [CH15, §4.3.1] (with slight generalization), and the reciprocity map rec_K induces an isomorphism

$$[-]_{m_0} : K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_{K,m_0}^\times \widehat{\mathbb{Q}}^\times \xrightarrow{\sim} G_{m_0} = \text{Gal}(H(m_0)/K).$$

Considering the following commutative diagram

$$\begin{array}{ccc} K^\times \backslash \widehat{K}^\times & \xrightarrow{\text{rec}_K} & \text{Gal}(K^{\text{ab}}/K) \\ \downarrow & & \downarrow \\ K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_{K,m_0}^\times \widehat{\mathbb{Q}}^\times & \xrightarrow[\simeq]{[-]_{m_0}} & G_{m_0} = \text{Gal}(H(m_0)/K) \\ \downarrow \psi_{m_0} & & \\ B^\times \backslash \widehat{B}^\times / \widehat{R}^\times & \xrightarrow{f} & \mathcal{O}, \end{array}$$

we define an element in group ring $\mathcal{O}[G_{m_0}]$ depending on f and ψ by

$$\tilde{\theta}_{m_0}(f) := \sum_a f(\psi_{m_0}(a)) [a]_{m_0} \in \mathcal{O}[G_{m_0}]$$

where a runs over $K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_{K,m_0}^\times \widehat{\mathbb{Q}}^\times$.

Remark 3.2. The map x_{m_0} in [CH15, §4.3.1] corresponds to $\psi_{p^{m_0}}$ in our setting. The image $\psi_{m_0}(a)$ is called a Gross points of conductor m_0 . If $K(p^\infty)/K$ is totally ramified, then the map ψ_{p^r} becomes much simpler as in [DI08, §2.2].

For $m = \prod q_i^{r_i}$, write $m_0 = \prod q_i^{r_i+1}$. We define the **theta element** $\theta_m(f)$ **attached to f and ψ** by the image of $\tilde{\theta}_{m_0}(f)$ of the natural projection $\mathcal{O}[G_{m_0}] \rightarrow \mathcal{O}[\Gamma_m]$ and **its involution** $\theta_m^*(f)$ by the image of $\theta_m(f)$ of the involution map $\mathcal{O}[\Gamma_m] \rightarrow \mathcal{O}[\Gamma_m]$ defined by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma_m$.

Remark 3.3. Theta elements $\theta_m(f)$ and $\theta_m^*(f)$ are well-defined only up to multiplication by Γ_m and a unit in \mathcal{O} .

We define the **Bertolini-Darmon element** $L_p(K(m), f)$ by

$$L_p(K(m), f) := \theta_m(f) \cdot \theta_m^*(f) \in \mathcal{O}[\Gamma_m],$$

and it is well-defined up to a unit in \mathcal{O} .

Remark 3.4 (on the interpolation formula). See [Hum17, Theorem A] for the interpolation formula for ring class characters of general conductors.

Let f_α be the p -stabilization of f with the unit U_p -eigenvalue $\alpha = \alpha_p(f)$. Let

$$\text{cores}_{r-1}^r : \mathcal{O}[\Gamma_{m'p^{r-1}}] \rightarrow \mathcal{O}[\Gamma_{m'p^r}]$$

be the corestriction map defined by $g \mapsto \sum_h hg$ where h runs over $\Gamma_{m'p^r}/\Gamma_{m'p^{r-1}}$ for all $r \geq 1$. We define the **p -stabilized theta element** $\theta_{m'p^r}(f_\alpha)$ by

$$\theta_{m'p^r}(f_\alpha) := \frac{1}{\alpha^r} \left(\theta_{m'p^r}(f) - \frac{1}{\alpha} \text{cores}_{r-1}^r \theta_{m'p^{r-1}}(f) \right)$$

for $r \geq 1$. Then $\{\theta_{m'p^r}(f_\alpha)\}_r$ forms a norm-compatible sequence and the inverse limit $\theta_{m'p^\infty}(f_\alpha)$ with respect to r is an element in $\mathcal{O}[[\Gamma_{m'p^\infty}]]$. In the same manner, we define the **generalized p -adic L -function** $L_p(K(m'p^\infty), f_\alpha)$ by

$$L_p(K(m'p^\infty), f_\alpha) = \theta_{m'p^\infty}(f_\alpha) \cdot \theta_{m'p^\infty}^*(f_\alpha) \in \mathcal{O}[[\Gamma_{m'p^\infty}]].$$

Remark 3.5. Note that $\text{cores}_{r-1}^r(a) = \Phi_{p^r}(\gamma_{p^r}) \cdot a \in \mathcal{O}[\Gamma_{m'p^r}]$ for $a \in \mathcal{O}[\Gamma_{m'p^{r-1}}]$. Moreover, $\Phi_{p^r}(\gamma_{p^r}) \cdot a = \Phi_{p^r}(\gamma_{p^r}) \cdot \tilde{a}$ for any lift \tilde{a} of a to $\mathcal{O}[\Gamma_{m'p^r}]$.

4. FITTING IDEALS, CHARACTER CONVENTION, AND THE ENHANCED ISOTYPIC DECOMPOSITION

4.1. Fitting ideals. Let A be a Noetherian ring. Let X be a finitely generated A -module, so it is of finite presentation. Fix a presentation

$$A^s \xrightarrow{h} A^r \longrightarrow X \longrightarrow 0$$

where $h \in M_{r \times s}(A)$.

Definition 4.1 (Fitting ideals). The **Fitting ideal** $\text{Fitt}_A(X)$ of X over A is the ideal of A generated by the determinant of the $r \times r$ -minors of the matrix h .

Remark 4.2. The Fitting ideal is independent of the choice of a presentation of X (so h).

Lemma 4.3 (Exact sequences; [MW84, 9, Appendix]). *Let $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ be an exact sequence of A -modules. Then $\text{Fitt}_A(X_1) \cdot \text{Fitt}_A(X_3) \subseteq \text{Fitt}_A(X_2)$.*

Lemma 4.4 (Projective limit; [MW84, 10, Appendix]). *Let A be a complete Noetherian local ring, X an A -module of finite presentation, which can be defined by the projective limit of quotient R -modules:*

$$X \simeq \varprojlim_n X_n$$

for $n \in \mathbb{N}$. Then $\text{Fitt}_A(X) = \bigcap_{n=1}^{\infty} \text{Fitt}_A(X_n)$.

Lemma 4.5 (Base change; [SU14, §3.1.5]).

(1) *For any Noetherian A -algebra B ,*

$$\text{Fitt}_B(X \otimes_A B) = \text{Fitt}_A(X)B$$

in B .

(2) *In particular, if $I \subset A$ is an ideal, then*

$$\text{Fitt}_{A/I}(X/IX) = \text{Fitt}_A(X) \pmod{I}$$

in A/I .

4.2. Convention on characters. Since we use various characters in this article, we fix the convention here. Consider three characters

$$\begin{aligned} \chi &: \Gamma_{m'p^\infty} \rightarrow \overline{\mathbb{Q}}_p^\times \\ \omega &: \Gamma_{m'} \rightarrow \overline{\mathbb{Q}}_p^\times \\ \varphi &: \Gamma_{m'p^\infty} \rightarrow \overline{\mathbb{Q}}_p^\times \end{aligned}$$

where

- χ is a ring class character of finite order used in §1.5,
- ω is a character on the prime-to- p part $\Gamma_{m'}$ used in §4.3 and §14, and
- φ is a character used in §15 for the specialization.

We recall the \mathbb{Q}_p -conjugacy relation of the characters in [Kur03, §1.4].

Definition 4.6 (Conjugacy relation of characters over \mathbb{Q}_p). We say that two characters ω_1 and ω_2 are **\mathbb{Q}_p -conjugate** if $\omega_1 = \sigma \omega_2$ for some $\sigma \in G_{\mathbb{Q}_p}$.

For any (p -adic) ring \mathcal{O}' of characteristic zero, let $\mathcal{O}'_?$ be the \mathcal{O}' -algebra generated by the values of $? \in \{\chi, \omega, \varphi\}$. Using these rings, we regard character $?$ as a $\mathcal{O}'_?$ -valued character. We also denote a uniformizer $\mathcal{O}'_?$ by $\varpi'_?$, the residue field by $\mathbb{F}'_?$, and $\mathcal{O}'_{?,n} := \mathcal{O}'_?/(\varpi'_?)^n$. Note that all \mathcal{O}_χ , \mathcal{O}_ω , and \mathcal{O}_φ are discrete valuation rings of characteristic zero. Also, they admit the action of $\Gamma_{m'p^\infty}$ or $\Gamma_{m'}$ via the characters.

Remark 4.7. Although χ and φ are same characters, we use different notation since their roles are completely different. The character φ is only used as the map of group rings $\mathcal{O}[[\Gamma_{m'p^\infty}]] \rightarrow \mathcal{O}_\varphi$.

Let M be an $\mathcal{O}[[\Gamma_{m'p^\infty}]]$ -module where \mathcal{O} is any p -adic ring of characteristic zero. Then we define the character-isotypic quotients

$$\begin{aligned} M_\chi &:= M \otimes_\chi \mathcal{O}_\chi \\ M_\varphi &:= M \otimes_\varphi \mathcal{O}_\varphi \end{aligned}$$

where the tensor products are taken over $\mathcal{O}[\Gamma_{m'p^\infty}]$ via χ and φ , respectively, and

$$M_\omega := M \otimes_\omega \mathcal{O}_\omega$$

where the tensor products are taken over $\mathcal{O}[\Gamma_{m'}]$ via ω . We call $M_?$ the **?-isotypic quotient of M** or the **specialization of M via $?$** $\in \{\chi, \omega, \varphi\}$.

4.3. The enhanced isotypic decomposition. We recall an *enhanced* version of the isotypic decomposition of modules over group rings following [Kur03, §1.3, 1.4]. Here we use isotypic quotients not isotypic subspaces.

Let $m = m'p^r$ be an integer with $(m', p) = 1$. Let $\omega : \Gamma_{m'} \rightarrow \mathcal{O}_\omega$ be a character as in §4.2. Then we have decomposition

$$\mathcal{O}[\Gamma_{m'}] = \oplus_\omega \mathcal{O}_\omega$$

where ω runs over $\text{Hom}(\Gamma_{m'}, \overline{\mathbb{Q}}_p^\times)$ up to the \mathbb{Q}_p -conjugation. The existence of the decomposition essentially comes from the fact m' is invertible in \mathcal{O} . The decomposition works even over $\mathcal{O}_n = \mathcal{O}/\varpi^n$ for the same reason.

Remark 4.8. In this article, \oplus_ω always means the direct sum over $\omega \in \text{Hom}(\Gamma_{m'}, \overline{\mathbb{Q}}_p^\times)$ up to the \mathbb{Q}_p -conjugation.

Then we have the following enhanced isotypic decomposition

$$\begin{aligned} M &= M \otimes_{\mathcal{O}[\Gamma_{m'p^\infty}]} \mathcal{O}[\Gamma_{m'p^\infty}] \\ &= M \otimes_{\mathcal{O}[\Gamma_{m'p^\infty}]} \mathcal{O}[\Gamma_{m'} \times \Gamma_{p^\infty}] \\ &= M \otimes_{\mathcal{O}[\Gamma_{m'p^\infty}]} (\oplus_\omega \mathcal{O}_\omega[\Gamma_{p^\infty}]) \\ &= \oplus_\omega \left(M \otimes_{\mathcal{O}[\Gamma_{m'p^\infty}]} \mathcal{O}_\omega[\Gamma_{p^\infty}] \right) \\ &= \oplus_\omega (M \otimes_\omega \mathcal{O}_\omega). \end{aligned}$$

Remark 4.9. It is equivalent to the classical isotypic decomposition if \mathcal{O} contains all the values of all the inequivalent characters. The *enhanced* decomposition removes the rationality question by nature.

Lemma 4.10. *Let M be a $\mathcal{O}[\Gamma_{m'p^\infty}]$ -module. If M_ω is free of rank s over $\mathcal{O}_\omega[\Gamma_{p^\infty}]$ for each $\omega \in \text{Hom}(\Gamma_{m'}, \overline{\mathbb{Q}}_p^\times)$ up to the \mathbb{Q}_p -conjugation, then M is free of rank s over $\mathcal{O}[\Gamma_{m'p^\infty}]$.*

Proof. It immediately follows from the enhanced isotypic decomposition:

$$\begin{aligned} M &\simeq \oplus_\omega M_\omega \\ &\simeq \oplus_\omega (\mathcal{O}_\omega[\Gamma_{p^\infty}])^s \\ &\simeq (\oplus_\omega \mathcal{O}_\omega[\Gamma_{p^\infty}])^s \\ &\simeq (\mathcal{O}[\Gamma_{m'p^\infty}])^s. \end{aligned}$$

□

5. GALOIS COHOMOLOGY AND SELMER GROUPS

We recall the language of Galois cohomology following [CH15, §1].

5.1. Galois cohomology. Let L be an algebraic extension of \mathbb{Q} . For a discrete $G_{\mathbb{Q}}$ -module M , we put

$$\begin{aligned} H^1(L_\ell, M) &:= \bigoplus_{\lambda|\ell} H^1(L_\lambda, M) \\ H^1(I_{L_\ell}, M) &:= \bigoplus_{\lambda|\ell} H^1(I_{L_\lambda}, M) \end{aligned}$$

where λ runs over all primes of L dividing ℓ and $I_{L_\lambda} \subset G_{L_\lambda}$ is the inertia subgroup. Denote the **restriction map at ℓ** by

$$\text{res}_\ell : H^1(L, M) \rightarrow H^1(L_\ell, M).$$

Define the **finite part** of $H^1(L_\ell, M)$ by

$$H_{\text{fin}}^1(L_\ell, M) := \ker (H^1(L_\ell, M) \rightarrow H^1(I_{L_\ell}, M))$$

and the **singular part** of $H^1(L_\ell, M)$ by

$$H_{\text{sing}}^1(L_\ell, M) := \frac{H^1(L_\ell, M)}{H_{\text{fin}}^1(L_\ell, M)}.$$

The natural map induced by the restriction

$$\partial_\ell : H^1(L, M) \rightarrow H_{\text{sing}}^1(L_\ell, M)$$

is called the **residue map**.

If $\kappa \in H^1(L, M)$ with $\partial_\ell(\kappa) = 0$, then the image of κ under res_ℓ lies in $H_{\text{fin}}^1(L_\ell, M)$. We write $v_\ell(\kappa) \in H_{\text{fin}}^1(L_\ell, M)$ for the image. For an n -admissible prime ℓ , $v_\ell(\kappa) \in H_{\text{fin}}^1(L_\ell, T_{f,n})$ can be extended to the case $\partial_\ell(\kappa) \neq 0$ due to the decomposition in Proposition 5.9.(5). See Definition 5.10.

5.2. Galois representations. Let ρ_f be the Galois representations attached to a newform $f \in S_2(\Gamma_0(N))^{\text{new}}$. Then ρ_f satisfies the following properties:

- (1) ρ_f is unramified outside pN ;
- (2) the restriction to $G_{\mathbb{Q}_p}$ is of the form $\begin{pmatrix} \chi_p^{-1}\varepsilon & * \\ 0 & \chi_p \end{pmatrix}$, where χ_p is unramified and $\chi_p(\text{Frob}_p) = \alpha$;
- (3) for all ℓ dividing N exactly, the restriction to $G_{\mathbb{Q}_\ell}$ is of the form $\begin{pmatrix} \pm\varepsilon & * \\ 0 & \pm 1 \end{pmatrix}$.

5.3. Ordinary local conditions. Let L/K be a finite extension. Let M be one of $V_f, T_f, A_f, T_{f,n}$, and $A_{f,n}$. For ℓ a prime factor of pN^- or an n -admissible prime (Definition 5.1), we define $G_{\mathbb{Q}_\ell}$ -invariant submodule $F_\ell^+ M$. If one have $F_\ell^+ V_f$, then one can define $F_\ell^+ M$ for other M as follows. Consider the exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & T_f & & F_\ell^+ V_f & & A_{f,n} \\ & & \downarrow \varpi^n & & \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & T_f & \xrightarrow{\text{inc}} & V_f & \xrightarrow{\pi} & A_f \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow & & \downarrow \varpi^n \\ & & T_{f,n} & & V_f / F_\ell^+ V_f & & A_f \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0. \end{array}$$

Then we define

$$\begin{aligned} F_\ell^+ T_f &:= \text{inc}^{-1} (F_\ell^+ V_f) \\ F_\ell^+ A_f &:= \pi (F_\ell^+ V_f) \\ F_\ell^+ T_{f,n} &:= \alpha (F_\ell^+ T_f) = \alpha (\text{inc}^{-1} (F_\ell^+ V_f)) \\ F_\ell^+ A_{f,n} &:= \beta^{-1} (F_\ell^+ A_f) = \beta^{-1} (\pi (F_\ell^+ V_f)). \end{aligned}$$

Using the above definitions, for $M = T_{f,n}$ and $A_{f,n}$, we define the **ordinary submodule** $H_{\text{ord}}^1(L_\ell, M)$ of $H^1(L_\ell, M)$ by

$$H_{\text{ord}}^1(L_\ell, M) := \ker (H^1(L_\ell, M) \rightarrow H^1(L_\ell, M/F_\ell^+ M)).$$

Thus, it suffices to specify $F_\ell^+ V_f$ for ℓ a prime factor of pN^- or an n -admissible prime. We use §5.2 here.

- (1) ($\ell = p$) Let $F_p^+ V_f \subset V_f$ be the E -rank-one $G_{\mathbb{Q}_p}$ -subspace on which the inertia subgroup I_p acts via ε .
- (2) (ℓ dividing N^-) Let $F_\ell^+ V_f \subset V_f$ be the E -rank-one $G_{\mathbb{Q}_\ell}$ -subspace on which the inertia subgroup I_p acts via ε or $\varepsilon\tau_\ell$, where ε is the p -adic cyclotomic character and τ_ℓ is the non-trivial unramified quadratic character of $G_{\mathbb{Q}_\ell}$.

The case for n -admissible primes is defined as follows.

Definition 5.1 (n -admissible primes). A rational prime ℓ is called **n -admissible for f** if it satisfies the following conditions:

- (1) ℓ does not divide pN ;
- (2) ℓ is inert in K/\mathbb{Q} ;
- (3) $\ell^2 \not\equiv 1 \pmod{p}$;
- (4) $\epsilon_\ell a_\ell(f) \equiv \ell + 1 \pmod{\varpi^n}$ with $\epsilon_\ell = \pm 1$.

If ℓ is n -admissible, then V_f is unramified at ℓ and $(\text{Frob}_\ell - \epsilon_\ell)(\text{Frob}_\ell - \epsilon_\ell \ell) = 0$ in $A_{f,n}$ and $T_{f,n}$ for the Frobenius Frob_ℓ of $G_{\mathbb{Q}_\ell}$. We let

- $F_\ell^+ A_{f,n} \subset A_{f,n}$ be the unique \mathcal{O}_n -corank-one submodule on which Frob_ℓ acts via the multiplication by $\epsilon_\ell \ell$, and
- $F_\ell^+ T_{f,n} \subset T_{f,n}$ be the unique \mathcal{O}_n -rank-one submodule on which Frob_ℓ acts via the multiplication by $\epsilon_\ell \ell$.

Then one can define the ordinary submodules $H_{\text{ord}}^1(L_\ell, A_{f,n})$ and $H_{\text{ord}}^1(L_\ell, T_{f,n})$ similarly.

5.4. Residual Δ -ordinary S -relaxed Selmer groups. Let Δ be a square-free integer such that Δ/N^- is a product of n -admissible primes. Let S be a square-free integer with $(S, \Delta N^+ p) = 1$.

Definition 5.2 (Δ -ordinary S -relaxed Selmer groups). Let $M = A_{f,n}$ or $T_{f,n}$. We define **Δ -ordinary S -relaxed Selmer groups** $\text{Sel}_\Delta^S(L, M)$ **attached to data** (f, n, Δ, S) to be the group of elements s in $H^1(L, M)$ satisfying the following properties:

- (1) $\partial_q(s) = 0$ for all $q \nmid p\Delta S$.
- (2) $\text{res}_q(s) \in H_{\text{ord}}^1(L_q, M)$ for $q \mid p\Delta$.
- (3) $\text{res}_q(s)$ is arbitrary for $q \mid S$.

In other words, the following sequence is exact:

$$0 \longrightarrow \text{Sel}_\Delta^S(L, M) \longrightarrow H^1(L, M) \xrightarrow{\prod_{q \mid p} \text{res}_q} \prod_{q \mid p} \frac{H^1(L_q, M)}{H_{\text{ord}}^1(L_q, M)} \times \prod_{q \mid \Delta} \frac{H^1(L_q, M)}{H_{\text{ord}}^1(L_q, M)} \times \prod_{q \nmid p\Delta S} \frac{H^1(L_q, M)}{H_{\text{fin}}^1(L_q, M)}$$

where q runs over all places of L .

Remark 5.3. This definition of Selmer groups depends on both $A_{f,n}$ (or $T_{f,n}$) and Δ . Note that Δ may not be recovered from $A_{f,n}$ (or $T_{f,n}$).

5.5. p -adic minimal Selmer groups and the comparison with N^- -ordinary Selmer groups.

We define the **minimal Selmer groups of f over $K(m'p^\infty)$** by the following exact sequence:

$$0 \longrightarrow \text{Sel}(K(m'p^\infty), A_f) \longrightarrow H^1(K(m'p^\infty), A_f) \xrightarrow{\prod_{q \mid p} \text{res}_q} \prod_{q \mid p} \frac{H^1(K(m'p^\infty)_q, A_f)}{H_{\text{ord}}^1(K(m'p^\infty)_q, A_f)} \times \prod_{q \nmid p} H^1(K(m'p^\infty)_q, A_f),$$

and we can also define the residual minimal Selmer group $\text{Sel}(K(m'p^\infty), A_{f,n})$ similarly (c.f. [EPW06, §4.2]). See also [PW11, §3.1].

Remark 5.4. These minimal Selmer groups are closer to the classical Selmer groups of elliptic curves than the N^- -ordinary Selmer groups. Also, these Selmer groups are more relevant to the context of anticyclotomic Iwasawa theory for modular forms than Greenberg Selmer groups as in [PW11, §3.1] due to the existence of primes which splits completely in the infinite extension.

Let Σ be the finite set of places of K containing the places dividing $Nm'p^\infty$, and K_Σ be the maximal extension of K unramified outside Σ . Then we have

$$\begin{aligned} 0 \longrightarrow \text{Sel}(K(m'p^\infty), A_f)[\varpi^n] &\longrightarrow H^1(K_\Sigma/K(m'p^\infty), A_f)[\varpi^n] \\ &\parallel \text{Lemma 14.3} \\ 0 \longrightarrow \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) &\longrightarrow H^1(K_\Sigma/K(m'p^\infty), A_{f,n}) \end{aligned}$$

since the residual representation is irreducible. Moreover, the local conditions at primes dividing N^- of minimal Selmer groups and N^- -ordinary Selmer groups coincide since such primes split completely in $K(m'p^\infty)/K$. Thus, we have the following sequence

$$0 \longrightarrow \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) \longrightarrow \text{Sel}(K(m'p^\infty), A_f)[\varpi^n] \longrightarrow \prod_w A_f^{G_{K(m'p^\infty)_w}} / \varpi^n A_f^{G_{K(m'p^\infty)_w}}$$

where w runs over primes of $K(m'p^\infty)$ dividing N^+ . Also, since we have the exact sequence

$$0 \longrightarrow \text{Sel}(K(m'p^\infty), A_{f,n}) \longrightarrow \text{Sel}(K(m'p^\infty), A_f)[\varpi^n] \longrightarrow \prod_w A_f^{G_{K(m'p^\infty)_w}} / \varpi^n A_f^{G_{K(m'p^\infty)_w}}$$

where w runs over primes of $K(m'p^\infty)$ dividing N ([EPW06, Corollary 4.2.5]), the inclusion

$$(1) \quad \text{Sel}(K(m'p^\infty), A_{f,n}) \subseteq \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) \subseteq \text{Sel}(K(m'p^\infty), A_f)[\varpi^n]$$

holds and the latter inclusion is of finite index. The index is bounded and independent of n . If f is N^+ -minimal, then

$$\text{Sel}(K(m'p^\infty), A_{f,n}) \subseteq \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) = \text{Sel}(K(m'p^\infty), A_f)[\varpi^n].$$

If f is N -minimal, then

$$\text{Sel}(K(m'p^\infty), A_{f,n}) = \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) = \text{Sel}(K(m'p^\infty), A_f)[\varpi^n].$$

See [EPW06, Lemma 4.1.2] for detail.

Proposition 5.5 (Comparison between Selmer groups I). *If f is N^+ -minimal, then*

$$\text{Sel}_{N^-}(K(m'p^\infty), A_f) := \varinjlim_n \text{Sel}_{N^-}(K(m'p^\infty), A_{f,n}) = \text{Sel}(K(m'p^\infty), A_f).$$

This comparison is enough for our purpose. However, we also prove the comparison statement without assuming N^+ -minimality with $m' = 1$. In other words, we give a detailed proof of [PW11, Proposition 3.6] and [CH15, Proposition 1.3].

Proposition 5.6. *If*

$$\text{Sel}_{N^-}(K(p^\infty), A_f)$$

is Λ -cotorsion with vanishing of μ -invariant, then

$$\text{Sel}(K(p^\infty), A_f)$$

is also Λ -cotorsion with vanishing of μ -invariant.

Proof. We have

$$\text{Sel}_{N^-}(K(p^\infty), A_f)[\varpi] = \text{Sel}_{N^-}(K(p^\infty), A_{f,1})$$

since the N^- -ordinary Selmer groups are defined as the injective limit of the residual Selmer groups. By assumption, $\text{Sel}_{N^-}(K(p^\infty), A_{f,1})$ is finite as in [KPW17, Proof of Corollary 2.3]. Since the inclusion

$$\text{Sel}_{N^-}(K(p^\infty), A_{f,1}) \subseteq \text{Sel}(K(p^\infty), A_f)[\varpi]$$

is of finite index (as in Inclusion (1)), $\text{Sel}(K(p^\infty), A_f)[\varpi]$ is also finite. Thus, we obtain the conclusion. \square

We still have the same conclusion without assuming the N^+ -minimality of f .

Corollary 5.7 (Comparison between Selmer groups II). *Assume the same conditions of Corollary 2.5,*

$$\text{Sel}_{N^-}(K(p^\infty), A_f) = \text{Sel}(K(p^\infty), A_f).$$

Proof. By Corollary 2.5, $\text{Sel}_{N^-}(K(p^\infty), A_f)$ is Λ_{p^∞} -cotorsion with vanishing of μ -invariant. By Proposition 5.6, $\text{Sel}(K(p^\infty), A_f)$ is also Λ_{p^∞} -cotorsion. We want to apply the work of Hachimori-Matsumo [HM00, Corollary] (Proposition 6.3) to minimal Selmer groups. Since the residual representation is irreducible and there exists a relevant Cassels-Tate pairing for minimal Selmer groups as in [Fla90], it suffices to check the control theorem works for minimal Selmer groups. Since we have the control theorem for N^- -ordinary Selmer groups (Lemma 14.2) and the index between N^- -ordinary Selmer groups and minimal Selmer groups is bounded independently on n , the control theorem for minimal Selmer groups holds. Thus, we obtain the conclusion. \square

Remark 5.8. It is not difficult to see that the minimal Selmer groups and the N^- -ordinary Selmer groups coincide under the N -minimality assumption. For the comparison over $K(p^\infty)$ between minimal Selmer groups and N^- -ordinary Selmer groups under Condition CR, see [PW11, Proposition 3.6] and [CH15, Proposition 1.3]. In the comparison argument over $K(p^\infty)$, we first take the direct limit $r \rightarrow \infty$ in $K(p^r)$ then take the direct limit $n \rightarrow \infty$ in $A_{f,n}$. Thus, $\text{Sel}_{N^-}(K(p^r), A_f)$ does not appear in the comparison over $K(p^\infty)$. Also, the comparison over $K(p^\infty)$ uses the theory of Λ -modules. In

order to compare them over $K(m)$ without any minimality condition, it seems that it involves certain Tamagawa exponents over $K(m)$ at ℓ dividing N^+ . See [PW11, Definition 3.3], [CH15, Corollary 2 and Corollary 6.15], and [Lun16, §1 and §4.2] for hints.

5.6. Local computation, local duality, and global reciprocity. We recall some computation and duality results of local cohomology over $K(m)$. All the proofs over $K(p^r)$ and $K(p^\infty)$ in [BD05] and [CH15] work for $K(m)$ in the exactly same way.

5.6.1. *Local computation.*

Proposition 5.9 (Local computation).

- (1) Let $\ell \neq p$ and be split in K/\mathbb{Q} . Let $m = m'p^r$ with $(m', p) = 1$. For $r \gg 0$, we have

$$\begin{aligned} H_{\text{sing}}^1(K(m)_\ell, T_{f,n}) &= 0 \\ H_{\text{fin}}^1(K(m)_\ell, A_{f,n}) &= 0. \end{aligned}$$

- (2) Let $\ell \neq p$ and be non-split in K/\mathbb{Q} .

$$\begin{aligned} H_{\text{sing}}^1(K(m)_\ell, T_{f,n}) &\simeq H_{\text{sing}}^1(K_\ell, T_{f,n}) \otimes_{\mathcal{O}_n} \mathcal{O}_n[\Gamma_m] \\ H_{\text{fin}}^1(K(m)_\ell, A_{f,n}) &\simeq \text{Hom}_{\mathcal{O}_n}(H_{\text{sing}}^1(K_\ell, T_{f,n}) \otimes_{\mathcal{O}_n} \mathcal{O}_n[\Gamma_m], E/\mathcal{O}) \end{aligned}$$

- (3) Let ℓ be an n -admissible prime.

$$\begin{aligned} H_{\text{sing}}^1(K_\ell, T_{f,n}) &\simeq \mathcal{O}_n \\ H_{\text{fin}}^1(K_\ell, A_{f,n}) &\simeq \mathcal{O}_n. \end{aligned}$$

- (4) Let ℓ be an n -admissible prime. Then $H_{\text{sing}}^1(K(m)_\ell, T_{f,n})$ and $H_{\text{fin}}^1(K(m)_\ell, T_{f,n})$ are free of rank one over $\mathcal{O}_n[\Gamma_m]$.

- (5) Let ℓ be an n -admissible prime. We have the decomposition

$$\begin{aligned} H^1(K(m)_\ell, T_{f,n}) &= H_{\text{fin}}^1(K(m)_\ell, T_{f,n}) \oplus H_{\text{ord}}^1(K(m)_\ell, T_{f,n}) \\ H^1(K(m)_\ell, A_{f,n}) &= H_{\text{fin}}^1(K(m)_\ell, A_{f,n}) \oplus H_{\text{ord}}^1(K(m)_\ell, A_{f,n}). \end{aligned}$$

Proof. Note that all the references below work for finite layers.

- (1) See [BD05, Lemma 2.4] and note that the same argument works for ℓ dividing N as in [CH15, Lemma 1.4.(1)].
- (2) See [BD05, Lemma 2.5] and note that the same argument works for ℓ dividing N as in [CH15, Lemma 1.4.(2)].
- (3) See [BD05, Lemma 2.6] and [CH15, Lemma 1.5.(1)].
- (4) See [BD05, Lemma 2.7] and [CH15, Lemma 1.5.(2)].
- (5) See [BD05, §4.1] and [CH15, Lemma 1.5.(3)].

□

Definition 5.10 (Localized projections at n -admissible primes). For $\kappa_m \in H^1(K(m), T_{f,n})$, $v_\ell(\kappa_m)$ is defined to be the projection of the restriction of κ_m at ℓ to $H_{\text{fin}}^1(K(m)_\ell, T_{f,n})$ according to the decomposition in Proposition 5.9.(5).

5.6.2. *Local duality.* We recall the Tate local duality at ℓ over $K(m)$, which is a perfect pairing

$$\langle -, - \rangle_\ell : H^1(K(m)_\ell, T_{f,n}) \times H^1(K(m)_\ell, A_{f,n}) \longrightarrow E/\mathcal{O}.$$

The pairing is compatible with action of $\mathcal{O}_n[\Gamma_m]$, so we have an isomorphism $H_{\text{fin}}^1(K(m)_\ell, T_{f,n}) \simeq H_{\text{fin}}^1(K(m)_\ell, A_{f,n})^\vee$ where $M^\vee = \text{Hom}_{\mathcal{O}}(M, E/\mathcal{O})$ as $\mathcal{O}_n[\Gamma_m]$ -modules.

Proposition 5.11 (Local dualities).

- (1) Let $\ell \neq p$. Then $H_{\text{fin}}^1(K(m)_\ell, T_{f,n})$ and $H_{\text{fin}}^1(K(m)_\ell, A_{f,n})$ are orthogonal complements under the pairing $\langle -, - \rangle_\ell$. In particular, $H_{\text{sing}}^1(K(m)_\ell, T_{f,n})$ and $H_{\text{fin}}^1(K(m)_\ell, A_{f,n})$ are the Pontryagin dual to each other.
- (2) Let ℓ be an n -admissible or ℓ divide pN^- . Assume Condition CR (for $\ell \mid N^-$) and Condition PO (for $\ell = p$). Then $H_{\text{ord}}^1(K(m)_\ell, T_{f,n})$ and $H_{\text{ord}}^1(K(m)_\ell, A_{f,n})$ are orthogonal complements under the pairing $\langle -, - \rangle_\ell$.

- (3) Let ℓ be an n -admissible. Then $H_{\text{fin}}^1(K(m)_\ell, T_{f,n})$ and $H_{\text{ord}}^1(K(m)_\ell, A_{f,n})$ are the Pontryagin dual to each other.

Proof. See [CH15, Proposition 1.6, Proposition 1.7, and Lemma 1.8] for proof with help of Lemma 5.12. Note that all the proofs work for $K(m)$. \square

Lemma 5.12 ([CH15, Lemma 1.8]). Assume Condition PO (Assumption 1.13). Then

$$H^0(K(m)_p, A_{f,n}/F^+A_{f,n}) = 0.$$

Proof. Let $F^-A_{f,n} := A_{f,n}/F^+A_{f,n}$. Let \wp be a place of $K(m)$ lying above p . Then the Frobenius $\text{Frob}_\wp \in \text{Gal}(K(m)/K)$ acts on $F^-A_{f,n}$ as $\alpha_p(f)^{r_p}$ where $\alpha_p(f)$ is the unit U_p -eigenvalue and $r_p = 1$ or 2 if p splits or is inert in K , respectively. Let m'' be the inertia degree of \wp in $K(m)/K$. If p is inert in K , then $m'' = 1$. Then

$$\begin{aligned} (F^-A_{f,n})^{G_{K(m)_\wp}} &= (F^-A_{f,n}) [(\text{Frob}_\wp)^{r_p m''} - 1] && \text{unramifiedness at } \wp \\ &= (F^-A_{f,n}) [\alpha_p(f)^{r_p m''} - 1] && \text{property of } \rho_f \\ &= \{0\} && \text{Condition PO.} \end{aligned}$$

\square

5.6.3. *Global reciprocity.* Let $\kappa_m \in H^1(K(m), T_{f,n})$ and $s_m \in H^1(K(m), A_{f,n})$. Via global class field theory, we have

$$\sum_q \langle \text{res}_q(\kappa_m), \text{res}_q(s_m) \rangle_q = 0$$

where q runs over all finite places of \mathbb{Q} .

Proposition 5.13 (Global reciprocity). Let $\kappa_m \in \text{Sel}_\Delta^S(K(m), T_{f,n})$ and $s_m \in \text{Sel}_\Delta(K(m), A_{f,n})$. Then

$$\sum_{q|S} \langle \partial_q(\kappa_m), v_q(s_m) \rangle_q = 0.$$

Proof. The local conditions of $\text{Sel}_\Delta^S(K(m), T_{f,n})$ and $\text{Sel}_\Delta(K(m), A_{f,n})$ are orthogonal at the primes not dividing S . \square

6. THE GENERALIZED DIVISIBILITY CRITERION

We recall the original divisibility criterion.

Proposition 6.1 ([BD05, Proposition 3.1]). Let X be a finitely generated Λ -module and $L \in \Lambda$. Suppose that $\varphi(L)$ belongs to $\text{Fitt}_{\mathcal{O}_\varphi}(X \otimes_\varphi \mathcal{O}_\varphi)$ for any homomorphism $\varphi : \Lambda \rightarrow \mathcal{O}_\varphi$ as in §4.2 and $\text{Fitt}_{\mathcal{O}_\varphi}(X \otimes_\varphi \mathcal{O}_\varphi)$ is the Fitting ideal of \mathcal{O}_φ -module of $X \otimes_\varphi \mathcal{O}_\varphi$. Then L belongs to the characteristic ideal of X .

This section is devoted to prove the following proposition.

Proposition 6.2 (The generalized divisibility criterion). Suppose that

$$\varphi(L) \in \varphi \left(\text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee) \right)$$

in \mathcal{O}_φ for any character $\varphi : \Lambda_{m'p^\infty} \rightarrow \mathcal{O}_\varphi$. Then

$$L \in \text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee).$$

The decomposition in §4.3 implies that the assumption for a character φ is equivalent to the statement

$$\overline{\varphi}(L_\omega) \in \overline{\varphi} \left(\text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee) \otimes_\omega \mathcal{O}_\omega \right)$$

holds for all character $\overline{\varphi} : \mathcal{O}_\omega[[\Gamma_{p^\infty}]] \rightarrow \mathcal{O}_\omega$ with $\varphi = \omega \times \overline{\varphi}$. Since Fitting ideals behave well under quotients and \mathcal{O}_ω is projective (so flat) over $\mathcal{O}[\Gamma_{m'}]$, we have

$$\begin{aligned} & \text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee) \otimes_\omega \mathcal{O}_\omega \\ (2) \quad &= \text{Fitt}_{\Lambda_{m'p^\infty}}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee) \mathcal{O}_\omega && \text{(flatness)} \\ &= \text{Fitt}_{\mathcal{O}_\omega[[\Gamma_{p^\infty}]]}(\text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega). && \text{(Lemma 4.5.(1))} \end{aligned}$$

Thus, the original divisibility criterion over $\mathcal{O}_\omega[\Gamma_{p^\infty}]$ (Proposition 6.1) implies that

$$(3) \quad L_\omega \in \text{char}_{\mathcal{O}_\omega[\Gamma_{p^\infty}]}(\text{Sel}_{N^-}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega).$$

In order to compare the characteristic ideal and the Fitting ideal, we need to prove the non-existence of non-trivial finite submodule of the dual Selmer modules over $\mathcal{O}_\omega[\Gamma_{p^\infty}]$ for each ω . We first recall the result over $\mathcal{O}[\Gamma_{p^\infty}]$.

Proposition 6.3 (Greenberg [Gre99, Proposition 4.14], Hachimori-Matsuno [HM00, Corollary]). *Suppose that*

- (1) $\text{Sel}_{N^-}(K(m'p^\infty), A_f)^\vee$ is $\mathcal{O}[\Gamma_{p^\infty}]$ -torsion, and
- (2) $H^0(K(m'), A_{f,1}) = 0$.

Then $\text{Sel}_{N^-}(K(m'p^\infty), A_f)^\vee$ has no non-trivial finite $\mathcal{O}[\Gamma_{p^\infty}]$ -submodule.

Remark 6.4. The statement in [HM00] uses the classical Selmer groups of elliptic curves. However, the argument in [HM00] works for any Selmer group satisfying the control theorem over Λ_{p^∞} if there exists a relevant Cassels-Tate pairing. Such a pairing is given in [Fla90]. Thus, it does not harm the validity of Proposition 6.3 although N^- -ordinary Selmer groups are used.

Corollary 6.5. *The module $\text{Sel}_{N^-}(K(m'p^\infty), A_f)^\vee$ has no non-trivial finite $\mathcal{O}[\Gamma_{p^\infty}]$ -submodule.*

Proof. The $(\omega = 1)$ -component of $L_p(K(m'p^\infty), f)$ is non-zero due to [Hum17, Theorems A and C]. Thus, the characteristic ideal of the dual Selmer group over $\mathcal{O}[\Gamma_{p^\infty}]$ is non-zero due to inclusion (3). Also, the residual representation is absolutely irreducible and $K(m')/\mathbb{Q}$ is Galois. \square

Indeed, this corollary correspond to the $\omega = 1$ -component. In general, we expect the following statement. In order to have Corollary 6.7, it suffice to show the following statement.

Lemma 6.6. *The module $\text{Sel}(K(m'p^\infty), A_f)^\vee$ has no non-trivial finite $\mathcal{O}[\Gamma_{m'p^\infty}]$ -submodule.*

Proof. Let C be a non-trivial finite $\mathcal{O}[\Gamma_{m'p^\infty}]$ -submodule $\text{Sel}(K(m'p^\infty), A_f)^\vee$. Let $c \in C$ be a non-zero element. Then the submodule of C generated by c over $\mathcal{O}[\Gamma_{p^\infty}]$ is the trivial module due to Corollary 6.5. Thus, the submodule of C generated by c over $\mathcal{O}[\Gamma_{m'}]$ must be non-trivial. In other words,

$$\mathcal{O}[\Gamma_{m'p^\infty}]c = \mathcal{O}[\Gamma_{m'}]c$$

since $\mathcal{O}[\Gamma_{p^\infty}]c = 0$. By decomposition $\mathcal{O}[\Gamma_{m'}]c = \bigoplus_\omega \mathcal{O}_\omega c_\omega$, one of $\mathcal{O}_\omega c_\omega$ should be non-trivial. (c_1 is trivial though.) However, $\mathcal{O}c$ is trivial since $\mathcal{O} \subset \Lambda_{p^\infty}$, so $\mathcal{O}c_\omega$ is trivial. Considering the natural surjective quotient map

$$\mathcal{O}[\Gamma_{m'}] \otimes_{\mathcal{O}} \mathcal{O}c_\omega \rightarrow \mathcal{O}_\omega c_\omega,$$

we get contradiction. \square

Corollary 6.7. *For each ω , the module $\text{Sel}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega$ has no nontrivial finite $\mathcal{O}_\omega[\Gamma_{p^\infty}]$ -submodule.*

Due to Corollary 6.7 and [Nuc10, Proposition 3], we have

$$L_\omega \in \text{Fitt}_{\mathcal{O}_\omega[\Gamma_{p^\infty}]}(\text{Sel}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega).$$

Considering the equality (2), we have

$$L \in \text{Fitt}_{\mathcal{O}[\Gamma_{m'p^\infty}]}(\text{Sel}(K(m'p^\infty), A_f)^\vee),$$

so we prove Proposition 6.2.

Remark 6.8. Note that the the finite layer analogue of Proposition 6.1 does not hold in general. The difficulty of the finite layer analogue is observed in [Lee15].

7. REDUCTION OF EULER SYSTEM DIVISIBILITIES TO MAZUR-TATE CONJECTURES

In this section, we show that the generalized Euler system divisibility (Theorem 1.17) and the exact control theorem (Lemma 14.2 with $S = 1$) imply Theorem 1.18 with the p -stabilized form f_α . The author learned this argument from Masato Kurihara.

Proposition 7.1. *Under the assumption of Theorem 1.17, we have*

$$L_p(K(m'p^r), f_\alpha) \in \text{Fitt}_{\mathcal{O}[\text{Gal}(K(m'p^r)/K)]}(\text{Sel}_{N-}(K(m'p^r), A_f)^\vee).$$

Proof. Write $X_{m'p^\infty} = \text{Sel}_{N-}(K(m'p^\infty), A_f)^\vee$ and $X_{m'p^r} = \text{Sel}_{N-}(K(m'p^r), A_f)^\vee$ where $(-)^\vee$ is the Pontryagin dual. Let $L_p(K(m'p^\infty), f_\alpha) = (L_p(K(m'p^r), f_\alpha))_r \in \Lambda_{m'p^\infty}$ be the generalized p -adic L -function defined in §3. Theorem 1.17 implies that

$$L_p(K(m'p^\infty), f_\alpha) \in \text{Fitt}_{\Lambda_{m'p^\infty}}(X_{m'p^\infty}).$$

By the exact control theorem (Lemma 14.2 with $S = 1$), we have

$$\text{Sel}_{N-}(K(m'p^\infty), A_f)^{(\Gamma_{m'p^\infty})^{p^r}} \simeq \text{Sel}_{N-}(K(m'p^r), A_f),$$

and the corresponding dual statement is

$$(X_{m'p^\infty})_{(\Gamma_{m'p^\infty})^{p^r}} \simeq X_{m'p^r}.$$

Let γ be a topological generator of $\Gamma_{m'p^\infty}$ and I_r be the ideal of $\Lambda_{m'p^\infty}$ generated by $\gamma^{p^r} - 1$. By Lemma 4.5.(2) and the above isomorphism, we have

$$\text{Fitt}_{\Lambda_{m'p^\infty}}(X_{m'p^\infty}) \pmod{I_r} = \text{Fitt}_{\mathcal{O}[\Gamma_{m'p^r}]}(X_{m'p^r}).$$

Thus, if we take mod I_r reduction of the inclusion

$$L_p(K(m'p^\infty), f_\alpha) = (L_p(K(m'p^r), f_\alpha))_r \in \text{Fitt}_{\Lambda_{m'p^\infty}}(X_{m'p^\infty}),$$

then we obtain

$$L_p(K(m'p^r), f_\alpha) \in \text{Fitt}_{\mathcal{O}[\Gamma_{m'p^r}]}(X_{m'p^r}).$$

□

8. p -STABILIZATION AND TAME EXCEPTIONAL ZEROES

In this section, we deduce the non- p -stabilized version of the Mazur-Tate conjecture from the p -stabilized version under a condition on the p -adic multiplier (Assumption 1.13).

8.1. p -ADIC MULTIPLIERS.

8.1.1. *p splits in K .* Suppose that p splits into $\mathfrak{p}\bar{\mathfrak{p}}$ in K . Then we define the p -adic multiplier $e_p(f, K)$ by

$$e_p(f, K) := (1 - \frac{1}{\alpha}\text{Fr}_{\mathfrak{p}})(1 - \frac{1}{\alpha}\text{Fr}_{\bar{\mathfrak{p}}}) \in \mathcal{O}[\Gamma_{m'}]$$

where $\text{Fr}_{\mathfrak{p}}, \text{Fr}_{\bar{\mathfrak{p}}}$ are the geometric Frobenii in $\Gamma_{m'} = \text{Gal}(K(m')/K)$. Then Condition PO (Assumption 1.13 for splitting p) directly implies that

$$e_p(f, K) \in \mathcal{O}[\Gamma_{m'}]^\times.$$

8.1.2. *p is inert in K .* Suppose that p is inert in K . Then

$$e_p(f, K) := (1 - \frac{1}{\alpha^2})$$

and it is also invertible in \mathcal{O} due to Condition PO (Assumption 1.13 for inert p).

8.2. Reduction via induction. We prove the following statement.

Proposition 8.1. *Under Condition PO, $\theta_{m'p^r}(f)$ is a multiple of $\theta_{m'p^r}(f_\alpha)$ by an element of $\mathcal{O}[\Gamma_{p^r}]$, i.e.*

$$\theta_{m'p^r}(f) = c \cdot \theta_{m'p^r}(f_\alpha)$$

where $c \in \mathcal{O}[\Gamma_{m'p^r}]$.

The following corollary shows that the Mazur-Tate conjecture with f_α implies the Mazur-Tate conjecture with f .

Corollary 8.2. *Let \mathfrak{a} be an ideal of $\mathcal{O}[\Gamma_{m'p^r}]$. If $\theta_{m'p^r}(f_\alpha) \in \mathfrak{a}$, then $\theta_{m'p^r}(f) \in \mathfrak{a}$.*

We prove the proposition via induction on r .

Proof. Considering the norm compatibility of the p -stabilized theta elements and the interpolation formula at the characters of prime-to- p conductor, we define

$$\theta_{m'}(f_\alpha) := e_p(f, K) \cdot \theta_{m'}(f)$$

where $\theta_{m'}(f_\alpha)$ is the image of $\theta_{m'p^r}(f_\alpha)$ in $\mathcal{O}[\Gamma_{m'}]$ for any $r \geq 1$. We assume that $e_p(f, K)$ is invertible in $\mathcal{O}[\Gamma_{m'}]$.

$$\begin{aligned} \theta_{m'p}(f_\alpha) &= \frac{1}{\alpha} \left(\theta_{m'p}(f) - \frac{1}{\alpha} \cdot \text{cores}_0^1(\theta_{m'}(f)) \right) \\ &= \frac{1}{\alpha} \left(\theta_{m'p}(f) - \frac{1}{\alpha} \cdot \text{cores}_0^1(e_p(f, K)^{-1} \cdot \theta_{m'}(f_\alpha)) \right) \\ &= \frac{1}{\alpha} \left(\theta_{m'p}(f) - \frac{1}{\alpha} \cdot \text{cores}_0^1(e_p(f, K)^{-1} \cdot \pi_0^1(\theta_{m'p}(f_\alpha))) \right) \\ &= \frac{1}{\alpha} \left(\theta_{m'p}(f) - \frac{1}{\alpha} \cdot \Phi_p(\gamma_p) \cdot c \cdot \theta_{m'p}(f_\alpha) \right) \end{aligned}$$

where $\pi_0^1 : \mathcal{O}[\Gamma_{m'p}] \rightarrow \mathcal{O}[\Gamma_{m'}]$ is the natural projection and c is a lift of $e_p(f, K)^{-1}$ in $\mathcal{O}[\Gamma_{m'p}]$, which is compatible with lifting $\theta_{m'p}(f_\alpha) \mapsto \theta_{m'}(f_\alpha)$ following Remark 3.5. Thus, we have

$$\theta_{m'p}(f) = \left(\alpha + \frac{1}{\alpha} \cdot \Phi_p(\gamma_p) \cdot c \right) \theta_{m'p}(f_\alpha).$$

Now suppose that

$$\theta_{m'p^{r-1}}(f) = b \cdot \theta_{m'p^{r-1}}(f_\alpha)$$

with $b \in \mathcal{O}[\Gamma_{m'p^{r-1}}]$. Then we have

$$\begin{aligned} \theta_{m'p^r}(f_\alpha) &= \frac{1}{\alpha^r} \left(\theta_{m'p^r}(f) - \frac{1}{\alpha} \cdot \text{cores}_{r-1}^r(\theta_{m'p^{r-1}}(f)) \right) \\ &= \frac{1}{\alpha^r} \left(\theta_{m'p^r}(f) - \frac{1}{\alpha} \cdot \text{cores}_{r-1}^r(b \cdot \theta_{m'p^{r-1}}(f_\alpha)) \right) \\ &= \frac{1}{\alpha^r} \left(\theta_{m'p^r}(f) - \frac{1}{\alpha} \cdot \text{cores}_{r-1}^r(b \cdot \pi_{r-1}^r(\theta_{m'p^r}(f_\alpha))) \right) \\ &= \frac{1}{\alpha^r} \left(\theta_{m'p^r}(f) - \frac{1}{\alpha} \cdot \Phi_{p^r}(\gamma_{p^r}) \cdot \tilde{b} \cdot \theta_{m'p^r}(f_\alpha) \right) \end{aligned}$$

where $\pi_{r-1}^r : \mathcal{O}[\Gamma_{m'p^r}] \rightarrow \mathcal{O}[\Gamma_{m'p^{r-1}}]$ is the natural projection and \tilde{b} is a lift of b in $\mathcal{O}[\Gamma_{m'p^r}]$, which is compatible with lifting $\theta_{m'p^r}(f_\alpha) \mapsto \theta_{m'p^{r-1}}(f_\alpha)$ following Remark 3.5. In the same manner, we have

$$\theta_{m'p^r}(f) = \left(\alpha^r + \frac{1}{\alpha} \cdot \Phi_{p^r}(\gamma_{p^r}) \cdot \tilde{b} \right) \theta_{m'p^r}(f_\alpha).$$

□

Even if Assumption 1.13 for splitting p fails for some ω , we are still able to obtain the following partial result.

Corollary 8.3. Assume that $e_p(f, K)_\omega$ is invertible in $\mathcal{O}_\omega[\Gamma_{p^r}]$. If

$$L(K(m'p^\infty), f_\alpha)_\omega \in \text{Fitt}_{\mathcal{O}_\omega[\Gamma_{p^\infty}]}(\text{Sel}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega),$$

then

$$L(K(m'p^\infty), f)_\omega \in \text{Fitt}_{\mathcal{O}_\omega[\Gamma_{p^\infty}]}(\text{Sel}(K(m'p^\infty), A_f)^\vee \otimes_\omega \mathcal{O}_\omega).$$

9. SHIMURA CURVES AND CM POINTS

We quickly review Shimura curves and CM points on them. See [CH15, §3] and [Nek12, §1] for detail.

9.1. Complex uniformization. Let B^Δ be the definite quaternion algebra over \mathbb{Q} of discriminant $\Delta = \Delta_B$. Let ℓ be a prime which is inert in K and is prime to Δ_B . Let $\mathcal{B}^{\Delta\ell}$ be the *indefinite* quaternion algebra over \mathbb{Q} of discriminant $\Delta\ell$. Fix a \mathbb{Q} -embedding $t^{\Delta\ell} : K \hookrightarrow \mathcal{B}^{\Delta\ell}$ and an isomorphism $\varphi_{B^\Delta, \mathcal{B}^{\Delta\ell}} : \widehat{B^\Delta}^{(\ell)} \simeq \widehat{\mathcal{B}^{\Delta\ell}}^{(\ell)}$. Let

- $U = U_{N^+}$ be the level structure of B^Δ corresponding to the unit group of an adelic Eichler order $\widehat{R}_{N^+}^\times = \widehat{R}^\times$ of level $\Gamma_0(N^+)$, and
- $U_0(\ell) = U_{N+\ell}$ be the level structure of B^Δ corresponding to the unit group of an adelic Eichler order $\widehat{R}_{N+\ell}^\times = \widehat{R_0(\ell)}^\times$ (contained in $\widehat{R}_{N^+}^\times$) of level $\Gamma_0(N+\ell)$.

Let

$$\mathcal{U} := \mathcal{U}_{N^+} = \varphi_{B^\Delta, \mathcal{B}^{\Delta\ell}} \left(U^{(\ell)} \right) \mathcal{O}_{\mathcal{B}^{\Delta\ell}}^\times$$

be the $\Gamma_0(N^+)$ -level structure of $\mathcal{B}^{\Delta\ell}$. We can also similarly define the $\Gamma_0(N^+) \cap \Gamma_1(p)$ -level structure $\mathcal{U}_{N^+, p}$.

Let $M_{\mathcal{U}}^{\Delta\ell} = M^{\Delta\ell}(N^+)$ be the Shimura curve over \mathbb{Q} attached to $\mathcal{B}^{\Delta\ell}$ of level \mathcal{U} . The complex uniformization of $M_{\mathcal{U}}^{\Delta\ell}$ is given by the double coset

$$M_{\mathcal{U}}^{\Delta\ell}(\mathbb{C}) = \mathcal{B}^{\Delta\ell, \times} \backslash \left((\mathbb{C} \setminus \mathbb{R}) \times \widehat{\mathcal{B}^{\Delta\ell}}^{\times} \right) / \mathcal{U}.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$ and $b \in \widehat{\mathcal{B}^{\Delta\ell}}^{\times}$, denote by $[z, b]_{\mathcal{U}}$ the point of $M_{\mathcal{U}}^{\Delta\ell}(\mathbb{C})$ represented by (z, b) .

9.2. ℓ -adic uniformization à la Cerednik-Drinfeld. Let $\mathfrak{M}^{\Delta\ell}(N^+)$ be an integral model of $M^{\Delta\ell}(N^+) \times_{\mathbb{Q}} \mathbb{Q}_\ell$ over \mathbb{Z}_ℓ . Let $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)$ be the formal completion of $\mathfrak{M}^{\Delta\ell}(N^+)$ along the special fiber. Let \mathfrak{h}_ℓ be the ℓ -adic upper half plane as a rigid analytic variety over \mathbb{Q}_ℓ , and $\widehat{\mathfrak{h}}_\ell$ be a natural formal model of \mathfrak{h}_ℓ . Then we have canonical identification

$$\widehat{\mathfrak{M}}^{\Delta\ell}(N^+) = B^{\Delta, \times} \backslash \left(\widehat{\mathfrak{h}}_\ell \widehat{\otimes}_{\mathbb{Z}_\ell} \widehat{\mathbb{Z}_\ell^{\text{ur}}} \right) \times \widehat{B^\Delta}^{(\ell), \times} / \widehat{R}^{(\ell), \times}.$$

Let $M^{\Delta\ell}(N^+)^{\text{an}}$ be the rigid analytification of $M^{\Delta\ell}(N^+) \times_{\mathbb{Q}} \mathbb{Q}_\ell$. Then $M^{\Delta\ell}(N^+)^{\text{an}}$ is the generic fiber of $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)$ and we have

$$M^{\Delta\ell}(N^+)^{\text{an}} = B^{\Delta, \times} \backslash \left(\mathfrak{h}_\ell \widehat{\otimes}_{\mathbb{Q}_\ell} \widehat{\mathbb{Q}_\ell^{\text{ur}}} \right) \times \widehat{B^\Delta}^{(\ell), \times} / \widehat{R}^{(\ell), \times}$$

and

$$M^{\Delta\ell}(N^+)(\mathbb{C}_\ell) = B^{\Delta, \times} \backslash \mathfrak{h}_\ell(\mathbb{C}_\ell) \times \widehat{B^\Delta}^{(\ell), \times} / \widehat{R}^{(\ell), \times}.$$

Remark 9.1. The quaternion algebra B^Δ is *definite*.

Remark 9.2 (on the notation). Although the formal completion of schemes, the finite adelization, and the completion of the maximal unramified extensions use the same notation, we expect that it does not make the readers confused.

9.3. Bad reduction of Shimura curves.

9.3.1. The dual graph of the fiber of the ℓ -adic upper half plane and the oriented Bruhat-Tits tree. Let $\mathcal{T}_\ell = \mathcal{V}(\mathcal{T}_\ell) \coprod \mathcal{E}(\mathcal{T}_\ell)$ be the dual graph of the special fiber of $\widehat{\mathfrak{h}}_\ell$, where $\mathcal{V}(\mathcal{T}_\ell)$ and $\mathcal{E}(\mathcal{T}_\ell)$ are the set of vertices and edges of \mathcal{T}_ℓ , respectively. Then \mathcal{T}_ℓ is the Bruhat-Tits tree of $B_\ell^\times / \mathbb{Q}_\ell^\times \simeq \text{PGL}_2(\mathbb{Q}_p)$.

Let $\vec{\mathcal{E}}(\mathcal{T}_\ell)$ be the set of oriented edges. Then we have the identifications $\mathcal{V}(\mathcal{T}_\ell) = B_\ell^\times / U_\ell \mathbb{Q}_\ell^\times$ and $\vec{\mathcal{E}}(\mathcal{T}_\ell) = B_\ell^\times / U_0(\ell)_\ell \mathbb{Q}_\ell^\times$ via the orbit-stabilizer theorem.

9.3.2. *The dual graph of the fiber of the formal model of the Shimura curves and the Bruhat-Tits tree.* Let

$$\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathcal{O}_{K_\ell}} := \widehat{\mathfrak{M}}^{\Delta\ell}(N^+) \times_{\mathbb{Z}_\ell} \mathcal{O}_{K_\ell}$$

be the base change of $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)$ over \mathcal{O}_{K_ℓ} and

$$\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathbb{F}_{\ell^2}} := \widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathcal{O}_{K_\ell}} \times_{\mathcal{O}_{K_\ell}} \mathbb{F}_{\ell^2}$$

be the semi-stable reduction fiber over \mathbb{F}_{ℓ^2} .

Let $\mathcal{G} = \mathcal{V}(\mathcal{G}) \coprod \mathcal{E}(\mathcal{G})$ be the dual graph of $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathbb{F}_{\ell^2}}$. The set $\mathcal{V}(\mathcal{G})$ of vertices of \mathcal{G} consists of the irreducible components of $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathbb{F}_{\ell^2}}$, and the set $\mathcal{E}(\mathcal{G})$ of edges of \mathcal{G} consists of the singular points of $\widehat{\mathfrak{M}}^{\Delta\ell}(N^+)_{\mathbb{F}_{\ell^2}}$.

9.3.3. *Reduction map and identifications.* We write B for B^Δ for convenience. Let

$$\text{red}_\ell : \mathfrak{M}^{\Delta\ell}(N^+)(\mathbb{C}_\ell) \rightarrow \mathcal{G} = \mathcal{V}(\mathcal{G}) \coprod \mathcal{E}(\mathcal{G})$$

be the reduction map modulo ℓ . The ℓ -adic uniformization result induces identifications

$$\begin{array}{ccc} \mathcal{V}(\mathcal{G}) & & \\ \parallel & & \\ B^\times \setminus \left(\mathcal{V}(\mathcal{G}) \times \mathbb{Z}/2\mathbb{Z} \times \widehat{B}^{(\ell),\times} / \widehat{R}^{(\ell),\times} \right) & & \\ \parallel & & \\ B^\times \setminus \left(B_\ell^\times / U_\ell \mathbb{Q}_\ell^\times \times \mathbb{Z}/2\mathbb{Z} \times \widehat{B}^{(\ell),\times} / \widehat{R}^{(\ell),\times} \right) & & B^\times (b_\ell U_\ell, j, b^{(\ell)} U^{(\ell)}) \\ \downarrow \simeq & & \downarrow \\ B^\times \setminus \widehat{B}^\times / \widehat{R}^\times \times \mathbb{Z}/2\mathbb{Z} & & ([b_\ell b^{(\ell)}]_U, j + \text{ord}_\ell(\text{nrd}(b_\ell))) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{E}(\mathcal{G}) & & \vec{\mathcal{E}}(\mathcal{G}) \\ \downarrow \simeq & & \downarrow \simeq \\ B^\times \setminus \widehat{B}^\times / \widehat{R}_0(\ell)^\times \times \{0\} & \hookrightarrow & B^\times \setminus \widehat{B}^\times / \widehat{R}_0(\ell)^\times \times \mathbb{Z}/2\mathbb{Z} \end{array}$$

where nrd is the reduced norm map.

9.4. Bad reduction of Jacobians of Shimura curves.

9.4.1. *Hecke action.* Let $J^{\Delta\ell}(N^+)$ be the Jacobian of the Shimura curve $M^{\Delta\ell}(N^+)$. If L/\mathbb{Q} is a field extension, let $\text{Div}^0 M^{\Delta\ell}(N^+)(L)$ be the group of divisors on $J^{\Delta\ell}(N^+)(L)$ of degree zero on each connected component of $M^{\Delta\ell}(N^+) \times_{\mathbb{Q}} L$. For $D \in \text{Div}^0 M^{\Delta\ell}(N^+)(L)$, write $cl(D) \in J^{\Delta\ell}(N^+)(L)$ for the point represented by D . Let $\mathbb{T}^{\Delta\ell,(\ell)}(N^+)$ be the ℓ -deprived Hecke algebra faithfully acting on $J^{\Delta\ell}(N^+)$ via Picard functoriality. The isomorphism $\varphi_{B^\Delta, \mathcal{B}^{\Delta\ell}} : \widehat{B}^{\Delta\ell} \simeq \widehat{\mathcal{B}}^{\Delta\ell}$ induces an isomorphism of the corresponding ℓ -deprived Hecke algebras $\varphi_{B^\Delta, \mathcal{B}^{\Delta\ell},*} : \mathbb{T}^{\Delta\ell,(\ell)}(\ell N^+) \simeq \mathbb{T}^{\Delta\ell,(\ell)}(N^+)$. We extend the map to the full Hecke algebras

$$\mathbb{T}^{\Delta\ell}(\ell N^+) \twoheadrightarrow \mathbb{T}^{\Delta\ell}(N^+) \rightarrow \text{End}(J^{\Delta\ell}(N^+)_{\mathbb{Q}})$$

by defining $U_\ell \mapsto \mathcal{R}_\ell \pi_\ell \mathcal{R}_\ell$ for some $\pi_\ell \in \mathcal{B}_\ell$ with $\text{nrd}(\pi_\ell) = \ell$ where \mathcal{R} is an Eichler order in $\mathcal{B}^{\Delta\ell}$ corresponding to \mathcal{U} .

Let $\mathfrak{J}^{\Delta\ell}(N^+)$ be the Néron model of $J^{\Delta\ell}(N^+)_{\mathbb{Q}_\ell}$ over \mathbb{Z}_ℓ . The universal property of Néron models induces an identification $\text{End}(J^{\Delta\ell}(N^+)_{\mathbb{Q}_\ell}) = \text{End}(\mathfrak{J}^{\Delta\ell}(N^+))$. Let $\mathfrak{J}_s := \mathfrak{J}^{\Delta\ell}(N^+) \times_{\mathbb{Z}_\ell} \mathbb{F}_{\ell^2}$ be the special fiber over \mathbb{F}_{ℓ^2} and \mathfrak{J}_s° be the connected component of \mathfrak{J}_s at the identity. Let $\Phi^{\Delta\ell}(N^+) := \mathfrak{J}_s / \mathfrak{J}_s^\circ$ be the group of connected components of \mathfrak{J}_s , which is an étale group scheme over \mathbb{F}_{ℓ^2} with action of $\mathbb{T}^{\Delta\ell}(\ell N^+)$ by the above argument. Let $\text{red}_\ell : J^{\Delta\ell}(N^+)(K_\ell) \rightarrow \Phi^{\Delta\ell}(N^+)$ be the reduction map.

9.4.2. *The graph description of the component group.* We describe $\Phi^{\Delta\ell}(N^+)$ in terms of the graph \mathcal{G} . See [BD05, §5.5], [Nek12, §1.6], and [CH15, §3.4] for detail. Let

$$\begin{aligned} s : \vec{\mathcal{E}}(\mathcal{G}) &\longrightarrow \mathcal{V}(\mathcal{G}) & e = (v_1 \rightarrow v_2) &\xrightarrow{s} s(e) = v_1 \\ t : \vec{\mathcal{E}}(\mathcal{G}) &\longrightarrow \mathcal{V}(\mathcal{G}) & e = (v_1 \rightarrow v_2) &\xrightarrow{t} t(e) = v_2 \end{aligned}$$

be the source and target maps. Using the identifications in §9.3.3, we induces maps (with the same notation)

$$\begin{array}{ccc} \vec{\mathcal{E}}(\mathcal{G}) & \xrightarrow{s} & \mathcal{V}(\mathcal{G}) \\ \uparrow & & \uparrow \\ \mathcal{E}(\mathcal{G}) & \xrightarrow{s} & B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \times \{0\} \end{array} \quad B^\times \widehat{bR_0(\ell)}^\times \xrightarrow{s} (B^\times \widehat{bR}^\times, 0)$$

and

$$\begin{array}{ccc} \vec{\mathcal{E}}(\mathcal{G}) & \xrightarrow{t} & \mathcal{V}(\mathcal{G}) \\ \uparrow & & \uparrow \\ \mathcal{E}(\mathcal{G}) & \xrightarrow{t} & B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \times \{1\} \end{array} \quad B^\times \widehat{bR_0(\ell)}^\times \xrightarrow{t} (B^\times \widehat{b\xi_\ell R}^\times, 1)$$

where $\xi_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_\ell) \simeq B_\ell^\times$. We induce the maps for the chain and cochain complexes of \mathcal{G} , which can be identified with degeneracy maps between the spaces of automorphic forms as follows:

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{E}(\mathcal{G})] & \xrightarrow{d_* = -s_* + t_*} & \mathbb{Z}[\mathcal{V}(\mathcal{G})] \\ \parallel & & \parallel \\ S_2^{N^-}(N^+, \mathbb{Z}) & \xrightarrow{-\alpha_* + \beta_*} & S_2^{N^-}(N^+, \mathbb{Z})^{\oplus 2} \end{array} \quad \begin{array}{ccc} \mathbb{Z}[\mathcal{V}(\mathcal{G})] & \xrightarrow{d^* = -s^* + t^*} & \mathbb{Z}[\mathcal{E}(\mathcal{G})] \\ \parallel & & \parallel \\ S_2^{N^-}(N^+, \mathbb{Z})^{\oplus 2} & \xrightarrow{-\alpha^* + \beta^*} & S_2^{N^-}(N^+, \mathbb{Z}) \end{array}$$

where $(\alpha^* f)(b) = f(b)$, $(\beta^* f)(b) = f(b\xi_\ell)$ are the injective degeneracy maps, and α_* and β_* are the corresponding trace maps as in [Nek12, §1.2.1].

Let $\mathbb{Z}[\mathcal{V}(\mathcal{G})]_0 := d_* (\mathbb{Z}[\mathcal{E}(\mathcal{G})]) \subseteq \mathbb{Z}[\mathcal{V}(\mathcal{G})]$. Following [Nek12, §1.6.5] (using [BD97, Appendix by Edixhoven]), we have a canonical isomorphism

$$\mathbb{Z}[\mathcal{E}(\mathcal{G})]/\mathrm{Im}(d^*) \xrightarrow[\simeq]{d_*} \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0/\mathrm{Im}(d_* \circ d^*) \simeq \Phi^{\Delta\ell}(N^+).$$

Let $D \in \mathrm{Div}^0(M^{\Delta\ell}(N^+)(K_\ell))$. Assume D has degree zero on each component of $M^{\Delta\ell}(N^+)_{\overline{K}_\ell}$. Extending D to a Cartier divisor \tilde{D} on $\mathfrak{M}^{\Delta\ell}(N) \times_{\mathbb{Z}_\ell} \mathcal{O}_{K_\ell}$ by taking closure of each effective ones, we define the map $cl_{\mathcal{V}}$ following [Nek12, §1.6.6]:

$$\begin{array}{ccccccc} cl_{\mathcal{V}} : \mathrm{Div}^0(M^{\Delta\ell}(N^+)(K_\ell)) & \xrightarrow{cl} & J^{\Delta\ell}(N^+)(K_\ell) & = & \mathfrak{J}^{\Delta\ell}(N^+)(\mathcal{O}_{K_\ell}) & \longrightarrow & \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0 \longrightarrow \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0/\mathrm{Im}(d_* \circ d^*) \\ D & \longmapsto & cl(D) = cl(\tilde{D}) & \longmapsto & \sum_{C \in \mathcal{V}(\mathcal{G})} (\tilde{D} \cdot C)C \end{array}$$

where $(\tilde{D} \cdot C)$ is the intersection number in $\mathfrak{M}^{\Delta\ell}(N^+)_{\mathcal{O}_{K_\ell}}$.

Then we have the following commutative diagram:

$$(4) \quad \begin{array}{ccc} \mathrm{Div}^0 M^{\Delta\ell}(N^+)(K_\ell) & \xrightarrow{cl} & J^{\Delta\ell}(N^+)(K_\ell) \\ \downarrow cl_{\mathcal{V}} & & \downarrow red_\ell \\ \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0/\mathrm{Im}(d_* \circ d^*) & \xrightarrow[\simeq]{} & \Phi^{\Delta\ell}(N^+). \end{array}$$

Remark 9.3. For the description using the space of automorphic forms, see [CH15, §3.5].

9.5. **CM points unramified at ℓ .** Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ be a point fixed by the image of K^\times in $\mathrm{GL}_2(\mathbb{R})$ under

$$K^\times \xrightarrow{t^{\Delta\ell}} \mathcal{B}^{\Delta\ell, \times} \xrightarrow{\otimes_{\mathbb{Q}} \mathbb{R}} \mathrm{GL}_2(\mathbb{R}).$$

The set of CM points by K unramified at ℓ on $M_{\mathcal{U}}^{\Delta\ell}$ is defined as

$$\begin{aligned} \text{CM}_{K,\ell}(M^{\Delta\ell}(N^+)) &:= \left\{ [z_0, b]_{\mathcal{U}} : b \in \widehat{\mathcal{B}^{\Delta\ell}}^{\times}, b_{\ell} = 1 \right\} \\ &\subset M^{\Delta\ell}(N^+)(K^{\text{ab}}) \end{aligned}$$

Shimura's reciprocity law implies that there exists an embedding

$$\iota_{\ell} : \text{CM}_{K,\ell}(M^{\Delta\ell}(N^+)) \subset M^{\Delta\ell}(N^+)(K_{\ell})$$

as in [Nek12, §1.8.3] and [CH15, §3.6].

Remark 9.4. The existence of the embedding ι_{ℓ} implies that ℓ splits completely in the field of definition of $\text{CM}_{K,\ell}(M^{\Delta\ell}(N^+))$ over K .

Let $\text{CM}_{K,\ell}^0(M^{\Delta\ell}(N^+)) \subset \text{Div}^0 M^{\Delta\ell}(N^+)(K_{\ell})$ be the subgroup generated by the degree zero divisors supported in $\text{CM}_{K,\ell}(M^{\Delta\ell}(N^+))$. Then we define the reduction map $\text{red}_{\mathcal{V}}$ by

$$\begin{aligned} \text{CM}_{K,\ell}^0(M^{\Delta\ell}(N^+)) &\xrightarrow{\text{red}_{\mathcal{V}}} \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0 \\ \sum_i n_i [z, b]_{\mathcal{U}} &\longmapsto \sum_i n_i \left[\varphi_{B^{\Delta}, \mathcal{B}^{\Delta\ell}}^{-1}(b) \right]_U \end{aligned}$$

and we have $\text{red}_{\mathcal{V}} \circ \iota_{\ell} = cl_{\mathcal{V}}$ where $cl_{\mathcal{V}}$ is defined in §9.4.2.

10. LEVEL RAISING MOD ϖ^n

Let $f_{\alpha} = \sum a_n(f)q^n \in S_2(\Gamma_0(N^+p\Delta))^{\text{new}}$ be the $\alpha = \alpha_p(f)$ -stabilization of f . Using the integral Jacquet-Langlands correspondence as in 3.1, we regard f_{α} as a modular form in $S_2^{\Delta}(N^+p)$. We define the \mathcal{O} -algebra homomorphism $\pi_{f_{\alpha}}$ associated to f_{α} by

$$\begin{aligned} \pi_{f_{\alpha}} : \mathbb{T}^{\Delta}(N^+p) &\rightarrow \mathcal{O} \\ T_q &\mapsto a_q(f) \text{ for } q \nmid N^+p\Delta \\ U_q &\mapsto a_q(f) \text{ for } q \mid N^+ \\ U_q &\mapsto \alpha_p(f) \text{ for } q = p \\ U_q &\mapsto \varepsilon_q(f) \text{ for } q \mid \Delta \\ \langle a \rangle &\mapsto 1. \end{aligned}$$

10.1. Level raising mod ϖ^n at one n -admissible prime: from definite to indefinite. We explain level raising mod ϖ^n at an n -admissible prime. See [BD05, Theorem 5.15], [CH15, §4.2], and [Lon12, §7.4.1] for detail.

Definition 10.1 (n -admissible forms). An n -admissible form $\mathcal{D} = (\Delta, g)$ is a pair consisting of

- a square-free integer Δ of an odd number of prime factors
- an \mathcal{O}_n -valued quaternionic eigenform $g \in S_2^{\Delta}(N^+p, \mathcal{O}_n)$

such that the following conditions hold:

- (1) $N^- \mid \Delta$ and every prime factor of Δ/N^- is n -admissible;
- (2) $g \not\equiv 0 \pmod{\varpi}$;
- (3) g is a $\mathbb{T}^{\Delta}(N^+p)$ -eigenform and $\pi_g \equiv \pi_{f_{\alpha}} \pmod{\varpi^n}$, where $\pi_g : \mathbb{T}^{\Delta}(N^+p) \rightarrow \mathcal{O}_n$ is the \mathcal{O} -algebra homomorphism induced by g .

We write $\mathcal{I}_g^{\Delta} := \ker(\pi_g)$.

Remark 10.2. Our n -admissible forms are slightly different from Chida-Hsieh's ones [CH15, Definition 4.1]. Level p^n in the Hecke algebra used in [CH15] is used for the reduction of weight via Hida theory, which is not used in our argument.

Remark 10.3. The concept of n -admissible forms allows us to remove all the rigid pair argument and the p -isolated condition in the former literature.

For an n -admissible form $\mathcal{D} = (\Delta, g)$, we define a surjective \mathcal{O} -linear map ψ_g by

$$\begin{array}{ccc} S_2^\Delta(N^+p, \mathcal{O})/\mathcal{I}_g^\Delta & \xrightarrow{\psi_g} & \mathcal{O}_n \\ h & \longmapsto & \langle g, h \rangle_U. \end{array}$$

Proposition 10.4 ([CH15, Proposition 4.2]). *Under Condition CR, ψ_g becomes an isomorphism*

$$\psi_g : S_2^\Delta(N^+, \mathcal{O})/\mathcal{I}_g^\Delta \simeq \mathcal{O}_n$$

Fix an n -admissible form $\mathcal{D} = (\Delta, g)$ and an n -admissible prime $\ell \nmid \Delta$ with $a_\ell(f) \equiv \varepsilon_\ell(\ell+1) \pmod{\varpi^n}$.

We first *formally* extend π_g to an \mathcal{O} -algebra homomorphism $\pi_{g'_\ell} : \mathbb{T}^\Delta(\ell N^+p) \rightarrow \mathcal{O}_n$ by defining $\pi_{g'_\ell}(U_\ell) = \varepsilon_\ell$. Since U_ℓ -eigenvalue is ε_ℓ modulo ϖ^n , the extended map $\pi_{g'_\ell}$ factors through the ℓ -new quotient $\mathbb{T}^{\Delta\ell}(N^+p)$ modulo ϖ^n . We define the \mathcal{O} -algebra homomorphism $\pi_{g_\ell} : \mathbb{T}^{\Delta\ell}(N^+p) \rightarrow \mathcal{O}_n$ by the following commutative diagram

$$\begin{array}{ccccc} & & \pi_{g'_\ell} & & \\ & \nearrow & & \searrow & \\ \mathbb{T}^\Delta(\ell N^+p) & \longrightarrow & \mathbb{T}^\Delta(N^+p) & \xrightarrow{\pi_g} & \mathcal{O}_n \\ & \searrow \text{\scriptsize ℓ-new quotient} & & \nearrow \pi_{g_\ell} & \\ & & \mathbb{T}^{\Delta\ell}(N^+p) & & \end{array}$$

and write $\mathcal{I}_{g_\ell}^{\Delta\ell} = \ker(\pi_{g_\ell})$.

Remark 10.5 (on the existence of $\Delta\ell$ -new form g_ℓ corresponding to π_{g_ℓ}). The existence of g_ℓ in $S_2^{\Delta\ell}(N^+p)$ itself is not a formal result at all because we also need to match up the relation between Hecke modules where the Hecke algebras act faithfully in the argument. In other words, we need to check that the Hecke eigensystem of π_{g_ℓ} arises in the Hecke module over \mathcal{O}_n where $\mathbb{T}^{\Delta\ell}(N^+p)$ acts faithfully. Note that Condition PO also applies to g_ℓ . Thus, the mod ϖ^n Hecke eigensystem of g_ℓ is also p -old. This is in fact tacitly used in [BD05, Theorem 5.15], whose proof uses Ribet's exact sequence extensively. One may use [DT94] instead. In fact, the existence of g_ℓ and Theorem 10.6 are the content of [BD05, Theorem 5.15] under Condition CR.

Let $\mathcal{B}^{\Delta\ell}$ be the indefinite quaternion algebra of discriminant $\Delta\ell$. Let $M^{\Delta\ell}(N^+p)$ be the Shimura curve attached to $\mathcal{B}^{\Delta\ell}$ of level $\mathcal{U}_0(p)$. Let $J^{\Delta\ell}(N^+p)$ be the Jacobian of $M^{\Delta\ell}(N^+p)$. Let $\Phi^{\Delta\ell}(N^+p)$ be the group of connected components of the special fibre of the Néron model of $J^{\Delta\ell}(N^+p)$ over \mathcal{O}_{K_ℓ} . Write $M_{\mathcal{O}} := M \otimes_{\mathbb{Z}} \mathcal{O}$.

Theorem 10.6 (Level raising mod ϖ^n ; [CH15, Theorem 4.3]). *Assume Condition CR. Then*

$$\Phi^{\Delta\ell}(N^+p)_{\mathcal{O}}/\mathcal{I}_{g_\ell}^{\Delta\ell} \simeq S_2^\Delta(N^+p)/\mathcal{I}_g^\Delta \simeq \mathcal{O}_n.$$

Let $\mathrm{Ta}_p(J^{\Delta\ell}(N^+p))$ be the p -adic Tate module of $J^{\Delta\ell}(N^+p)$.

Corollary 10.7 ([CH15, Corollary 4.4]). *Under Condition CR, we have*

$$\mathrm{Ta}_p(J^{\Delta\ell}(N^+p))_{\mathcal{O}}/\mathcal{I}_{g_\ell}^{\Delta\ell} \simeq T_{f,n}.$$

Remark 10.8. Corollary 10.7 can be regarded as another description of the Jochnowitz congruences in terms of Galois representations.

Consider the (enhanced) reduction map

$$\widetilde{\mathrm{red}}_\ell : J^{\Delta\ell}(N^+p)(K(m)) \rightarrow \Phi^{\Delta\ell}(N^+p)_{\mathcal{O}} \otimes \mathcal{O}_n[\Gamma_m]$$

defined by

$$\widetilde{\mathrm{red}}_\ell(D) := \sum_{\sigma \in \Gamma_m} \mathrm{red}_\ell(\iota_\ell(\sigma D)) \sigma$$

where $\iota_\ell : J^{\Delta\ell}(N^+p)(K(m)) \rightarrow J^{\Delta\ell}(N^+p)(K_\ell)$ is an embedding since ℓ splits completely in $K(m)/K$. This ι_ℓ coincides with the embedding with the same notation in §9.5.

The next theorem is used to prove the first explicit reciprocity law.

Theorem 10.9 ([CH15, Theorem 4.5]). *Assume Condition CR.*

(1) There is an isomorphism of \mathcal{O}_n -modules

$$\psi_g : \Phi^{\Delta_\ell}(N^+p)_{\mathcal{O}}/\mathcal{I}_{g_\ell}^{\Delta_\ell} \xrightarrow{\simeq} H_{\text{sing}}^1(K_\ell, T_{f,n})$$

which is canonical up to the choice of an isomorphism in Corollary 10.7.

(2) There is a commutative diagram

$$\begin{array}{ccc} J^{\Delta_\ell}(N^+p)(K(m))/\mathcal{I}_{g_\ell}^{\Delta_\ell} & \xrightarrow{\text{Kum}} & H^1(K(m), T_{f,n}) \\ \downarrow \widetilde{\text{red}}_\ell & & \downarrow \partial_\ell \\ \Phi^{\Delta_\ell}(N^+p)_{\mathcal{O}} \otimes \mathcal{O}_n[\Gamma_m] & \xrightarrow[\simeq]{\psi_g} & H_{\text{sing}}^1(K(m)_\ell, T_{f,n}). \end{array}$$

10.2. Ihara input. We recall two results: (a special case of) Ihara's lemma for weight 2 forms on Shimura curves and Ihara's work on rational points on Shimura curves over finite fields.

10.2.1. Ihara's lemma for Shimura curves. Let $q \geq 4$ be an integer dividing M^+ . Let \mathcal{B} be the indefinite quaternion algebra over \mathbb{Q} of discriminant $\Delta_{\mathcal{B}}$ and $\mathcal{V} = \widehat{\mathcal{R}}_{M^+/q,q}^\times$ be the open subgroup of $\widehat{\mathcal{B}}^\times$ representing the $\Gamma_0(M^+/q) \cap \Gamma_1(q)$ -level structure. Let $M_{\mathcal{V}}^{\Delta_{\mathcal{B}}} = M^{\Delta_{\mathcal{B}}}(N^+/q, q)$ be the corresponding Shimura curve. Let $\mathcal{L}_2(N^+/q, q, \mathbb{F}) := H_{\text{ét}}^1(M^{\Delta_{\mathcal{B}}}(M^+/q, q)_{\overline{\mathbb{Q}}}, \underline{\mathbb{F}})$ be the étale cohomology of the Shimura curve with \mathbb{F} -coefficients, where $\underline{\mathbb{F}}$ is the constant étale sheaf associated to \mathbb{F} . Let ℓ be a prime not dividing $p\Delta_{\mathcal{B}}$. Let $\mathfrak{m}_{g'_\ell} \subseteq \mathbb{T}^{\Delta_{\mathcal{B}}}(\ell M^+/q, q)$ be the maximal ideal containing the kernel of the map

$$\pi_{g'_\ell} : \mathbb{T}^{\Delta_{\mathcal{B}}}(\ell M^+/q, q) \rightarrow \mathbb{F}$$

as in §10.1 although the role of the definite and the indefinite changes. Then it is also ordinary and non-Eisenstein.

Theorem 10.10 (Ihara's lemma for Shimura curves; [DT94, Theorem 2]). *Let $q \geq 4$ be an integer dividing M^+ . Assume that p and ℓ do not divide M^+ . Then the map*

$$\alpha_\ell = 1 + \eta_\ell : (\mathcal{L}_2(M^+/q, q, \mathbb{F})^{\oplus 2})_{\mathfrak{m}_{g'_\ell}} \rightarrow (\mathcal{L}_2(\ell M^+/q, q, \mathbb{F}))_{\mathfrak{m}_{g'_\ell}}$$

is injective where η_ℓ is the degeneracy map at ℓ .

Remark 10.11. (1) The module $\mathcal{L}_2(M^+/q, q, \mathbb{F})^{\oplus 2}$ admits a $\mathbb{T}^{\Delta_{\mathcal{B}}}(\ell M^+/q, q)$ -module structure using the argument in [BD05, Proposition 5.8] or [CH15, §3.5].

(2) The original statement is given with \mathbb{F}_p -coefficients, but the extension of coefficients to \mathbb{F} does not change anything.

Now we put $M^+ = N^+p$ and $q = p$. Then N^+p does not satisfies the maximal condition at p on the level. However, the following lemma gives us a salvage.

Lemma 10.12 ([CH15, Lemma 5.3]). *Suppose that \mathfrak{m} satisfies Condition PO. Then*

$$\mathcal{L}_2(N^+, \mathbb{F})_{\mathfrak{m}} \simeq \mathcal{L}_2(N^+p, \mathbb{F})_{\mathfrak{m}} \simeq \mathcal{L}_2(N^+, p, \mathbb{F})_{\mathfrak{m}}.$$

Due to Lemma 10.12, we can remove the maximal condition on the level at p in Theorem 10.10.

Corollary 10.13. *Suppose that $\mathfrak{m}_{g'_\ell}$ satisfies Condition PO. Assume that p and ℓ do not divide N^+ . Then the map*

$$\alpha_\ell = 1 + \eta_\ell : (\mathcal{L}_2(N^+, p, \mathbb{F})^{\oplus 2})_{\mathfrak{m}_{g'_\ell}} \rightarrow (\mathcal{L}_2(\ell N^+, p, \mathbb{F}))_{\mathfrak{m}_{g'_\ell}}$$

is injective where η_ℓ is the degeneracy map at ℓ .

10.2.2. Rational points on Shimura curves over finite fields. Let \mathcal{B} be the indefinite quaternion algebra over \mathbb{Q} of discriminant $\Delta_{\mathcal{B}}$. Fix an embedding \mathcal{B} into $M_2(\mathbb{R})$, which is compatible with the convention in §9.1. Let $\mathcal{R} \subset \mathcal{B}$ be an Eichler order of level N^+p . Let Γ_{∞, N^+p} be the image of the group of reduced norm one elements in $\mathcal{R}[1/\ell]^\times/\{\pm 1\}$ in $\text{PSL}_2(\mathbb{R})$. Then $M^{\Delta_{\mathcal{B}}}(N^+)(\mathbb{C}) \simeq \Gamma_{\infty, N^+p} \backslash \mathfrak{h}$ as compact Riemann surfaces where \mathfrak{h} is the complex upper-half plane. See also [BD05, §5.1]. Let $\Gamma_{\infty, N^+, p} \subset \Gamma_{\infty, N^+p}$ be the finite index subgroup which is determined by $M^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{C}) \simeq \Gamma_{\infty, N^+, p} \backslash \mathfrak{h}$. Then $\Gamma_{\infty, N^+, p}$ is torsion-free. See [BD05, Proof of Proposition 9.2]. See also [DT94, Corollary 3, §4] for the geometric property of the torsion-freeness. Under this setting, we have the following statement.

Proposition 10.14 ([Iha99, (G), §3]). *Let ℓ be a prime not dividing $N^+p\Delta_{\mathcal{B}}$. Then we have*

$$\mathfrak{J}^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{F}_{\ell^2})/\mathfrak{J}^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{F}_{\ell^2})^{ss} \simeq (\Gamma_{\infty, N^+, p})^{ab}$$

where $\mathfrak{J}^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{F}_{\ell^2})^{ss} \subset \mathfrak{J}^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{F}_{\ell^2})$ is the subgroup generated by the divisors supported on the supersingular points in $\mathfrak{J}^{\Delta_{\mathcal{B}}}(N^+, p)(\mathbb{F}_{\ell^2})$ and $(\Gamma_{\infty, N^+, p})^{ab}$ is the abelianization of $\Gamma_{\infty, N^+, p}$.

10.3. Level raising mod ϖ^n at two n -admissible prime: from definite to definite passing through indefinite. We study the level raising at two n -admissible primes in detail. We focus on how “Ihara input” is used. In order to apply Ihara’s work, one needs to work with the level smaller than the $\Gamma_1(q)$ -level structure with $q \geq 4$. Note that the level-raised form exists only in the localized Hecke module. See [BD05, §9], [Lon12, §7.4.3] and [vO13, §11, Level raising at two primes] for detail. Also, see [CH15, §5.3] for the moduli description.

We first recall some definitions to draw the important commutative diagram, which exactly describes the level raising at two n -admissible primes. Let ℓ_1 and ℓ_2 be distinct n -admissible primes not dividing Δ . Let $\mathbb{T}^{\Delta_{\ell_1}}(N^+p)$ be the Hecke algebra acting on Shimura curve $X^{\Delta_{\ell_1}}(N^+p)$. Let $J^{\Delta_{\ell_1}}(N^+p)(K_{\ell_2})$ be the K_{ℓ_2} -points of the Jacobian of $X^{\Delta_{\ell_1}}(N^+p)$. Let $\mathfrak{J}^{\Delta_{\ell_1}}(N^+)(\mathbb{F}_{(\ell_2)^2})$ be the $\mathbb{F}_{(\ell_2)^2}$ -points of the reduction at ℓ_2 of the Néron model $\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)$ over $\mathcal{O}_{K_{\ell_2}}$. Let $\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss}$ be the subgroup generated by the divisors supported on the supersingular points in $\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})$. We first draw the following commutative diagram and explain the nontrivial part.

$$\begin{array}{ccc}
J^{\Delta_{\ell_1}}(N^+p)(K_{\ell_2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} & \xrightarrow{\text{Kum}_{\ell_2}} & H^1(K_{\ell_2}, \text{Ta}_p(J^{\Delta_{\ell_1}}(N^+p))/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}) \xrightarrow{\simeq} H^1(K_{\ell_2}, T_{f,n}) \\
\downarrow \text{red}_{\ell_2} \simeq & & \uparrow \simeq \downarrow v_{\ell_2} \\
\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} & & H_{\text{fin}}^1(K_{\ell_2}, T_{f,n}) \\
\uparrow \cup & & \downarrow \simeq \\
\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss}/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} & & \mathcal{O}_n \\
\uparrow & \nearrow g_{\ell_1 \ell_2}^* & \\
\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss} & & \\
\uparrow \simeq & & \\
\mathbb{Z}[B^{\Delta_{\ell_1 \ell_2}, \times} \setminus \widehat{B}^{\Delta_{\ell_1 \ell_2}, \times} / \widehat{R}_{N^+p}^{\times}] & &
\end{array}$$

Using the above diagram and the pairing in §3.1, we define the level-raised n -admissible form $\mathcal{D}_n^{\ell_2 \ell_2} = (\Delta_{\ell_1 \ell_2}, g_{\ell_1 \ell_2})$ by the equality

$$g_{\ell_1 \ell_2}^*(b) = \langle g_{\ell_1 \ell_2}, b \rangle_U.$$

Remark 10.15 (on the diagram).

- (1) The form $g_{\ell_1 \ell_2}$ is a mod ϖ^n $\mathbb{T}^{\Delta_{\ell_1 \ell_2}}(N^+p)$ -eigenform due to [BD05, Lemma 9.1].
- (2) The reduction map at $\mathbb{F}_{(\ell_2)^2}$

$$\text{red}_{\ell_2} : J^{\Delta_{\ell_1}}(N^+p)(K_{\ell_2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} \rightarrow \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}$$

is an isomorphism because $J^{\Delta_{\ell_1}}(N^+p)$ has good reduction at ℓ_2 .

- (3) The natural inclusion

$$H_{\text{fin}}^1(K_{\ell_2}, T_{f,n}) \hookrightarrow H^1(K_{\ell_2}, T_{f,n})$$

also becomes an isomorphism because $\text{Ta}_p(J^{\Delta_{\ell_1}}(N^+p))/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} \simeq T_{f,n}$ is unramified at ℓ_2 .

- (4) Since the image of the Kummer map at ℓ_2 of $J^{\Delta_{\ell_1}}(N^+p)(K_{\ell_2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}$ is $H_{\text{fin}}^1(K_{\ell_2}, T_{f,n})$, the Kummer map Kum_{ℓ_2} becomes surjective.

Following [BD05, Proposition 9.2], [Lon12, §7.4.3], and [vO13, Proposition 11.12], we give a detailed exposition of the following statement.

Proposition 10.16. *Under Condition Ihara, the composition*

$$J^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss} \rightarrow J^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss}/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}} \hookrightarrow J^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}$$

in the above diagram is surjective. Thus, $g_{\ell_1\ell_2}$ is surjective. If we further assume Condition CR, then $g_{\ell_1\ell_2}$ is determined uniquely up to \mathcal{O}_n^\times .

Remark 10.17. The surjectivity of $g_{\ell_1\ell_2}$ is essential because it ensures nonvanishing modulo ϖ .

The rest of this section is devoted to prove Proposition 10.16. We first recall the following result on trees.

Proposition 10.18 ([Ser03, Proposition 13, §II.2.8]). *Let \mathcal{T} be a tree and G be a group acting on \mathcal{T} . Let Σ_0 be the set of representatives of $\mathcal{V}(\mathcal{T})/G$ and Σ_1 the set of representatives of $\mathcal{E}(\mathcal{T})/G$. Let M be a G -module. Then we have an exact cohomology sequence*

$$\cdots \rightarrow H^i(G, M) \rightarrow \prod_{v \in \Sigma_0} H^i(G_v, M) \rightarrow \prod_{e \in \Sigma_1} H^i(G_e, M) \rightarrow H^{i+1}(G, M) \rightarrow \cdots$$

where $G_v = \text{Stab}_G(v)$ and $G_e = \text{Stab}_G(e)$.

We fix an embedding $\mathcal{B}^{\Delta_{\ell_1}}$ into $M_2(\mathbb{Q}_{\ell_2})$. Let Γ_{ℓ_2, N^+p} be the image of the group of reduced norm one elements in $\mathcal{R}_{N^+p}[1/\ell]^\times/\{\pm 1\}$ in $\text{PGL}_2(\mathbb{Q}_{\ell_2})$ and $\Gamma_{\ell_2, N^+, p} \subset \Gamma_{\ell_2, N^+p}$ be the similarly defined subgroup as in §10.2.2.

Remark 10.19. We have isomorphisms $\Gamma_{\ell_2, N^+p} \simeq \Gamma_{\infty, N^+p}$ and $\Gamma_{\ell_2, N^+, p} \simeq \Gamma_{\infty, N^+, p}$ as groups since both maps to $\text{GL}_2(\mathbb{Q}_{\ell_2})$ and $\text{GL}_2(\mathbb{R})$ are embeddings.

Then we obtain an action of Γ_{ℓ_2, N^+p} (so also of $\Gamma_{\ell_2, N^+, p}$) on the Bruhat-Tits tree \mathcal{T}_{ℓ_2} for $\text{PGL}_2(\mathbb{Q}_{\ell_2})$.

Let $v_0 \in \mathcal{V}(\mathcal{T}_{\ell_2})$ such that $(\Gamma_{\ell_2, N^+, p})_{v_0} := \text{Stab}_{\Gamma_{\ell_2, N^+, p}}(v_0) \subseteq \text{PSL}_2(\mathbb{Z}_{\ell_2})$. Let e_0 be the edge whose source is v_0 such that the stabilizer $(\Gamma_{\ell_2, N^+, p})_{e_0}$ is the subgroup of $(\Gamma_{\ell_2, N^+, p})_{v_0}$ consisting of the upper triangular elements modulo ℓ_2 . Write v_1 for the target of e_0 . Consider the identifications

$$\begin{aligned} (\Gamma_{\ell_2, N^+, p})_{v_0} &\simeq (\Gamma_{\infty, N^+, p})_{v_0} \subset \text{PSL}_2(\mathbb{R}) \\ (\Gamma_{\ell_2, N^+, p})_{e_0} &\simeq (\Gamma_{\infty, N^+, p})_{e_0} \subset \text{PSL}_2(\mathbb{R}). \end{aligned}$$

Then the discrete subgroups at the infinite place define the corresponding Shimura curves as follows:

- $(\Gamma_{\infty, N^+, p})_{v_0}$ defines Shimura curve $M^{\Delta_{\ell_1}}(N^+, p)$, and
- $(\Gamma_{\infty, N^+, p})_{e_0}$ defines Shimura curve $M^{\Delta_{\ell_1}}(\ell_2 N^+, p)$, respectively.

Applying Proposition 10.18 with $i = 1$, $M = \mathbb{F}$ and $G = \Gamma_{\ell_2, N^+, p}$, we have the following statement.

Corollary 10.20. *The sequence*

$$\text{Hom}(\Gamma_{\infty, N^+, p}, \mathbb{F}) \longrightarrow \text{Hom}\left((\Gamma_{\infty, N^+, p})_{v_0}, \mathbb{F}\right) \oplus \text{Hom}\left((\Gamma_{\infty, N^+, p})_{v_1}, \mathbb{F}\right) \xrightarrow{d} \text{Hom}\left((\Gamma_{\infty, N^+, p})_{e_0}, \mathbb{F}\right)$$

is exact.

By [Lin93, (3), §1], a canonical homomorphism $J^{\Delta_{\ell_1}}(N^+, p)(\mathbb{C}) \rightarrow \text{Hom}(\Gamma_{\infty, N^+, p}, S^1)$ is injective where S^1 is the unit circle in \mathbb{C} . Thus, we can identify the source and the target of d with the ϖ -torsions of the Jacobians of the corresponding Shimura curves. Also, we identify the ϖ -torsions of the Jacobians with the first Betti cohomologies of the corresponding Shimura curves with $\mathbb{F}(1)$ -coefficients, where $\mathbb{F}(1)$ is the Tate twist of \mathbb{F} . We identify the Betti cohomologies with mod ϖ étale cohomologies and take the inverse twist of the coefficients. Then the map d becomes α_{ℓ_2} as in Theorem 10.10. The following commutative diagram summarizes all the identifications.

$$\begin{array}{ccc} \text{Hom}(\Gamma_{\infty, N^+, p}, \mathbb{F}) & \longrightarrow & \text{Hom}\left((\Gamma_{\infty, N^+, p})_{v_0}, \mathbb{F}\right) \oplus \text{Hom}\left((\Gamma_{\infty, N^+, p})_{v_1}, \mathbb{F}\right) \xrightarrow{d} \text{Hom}\left((\Gamma_{\infty, N^+, p})_{e_0}, \mathbb{F}\right) \\ & & \parallel \\ & & J^{\Delta_{\ell_1}}(N^+, p)(\mathbb{C})[\varpi] \oplus J^{\Delta_{\ell_1}}(N^+, p)(\mathbb{C})[\varpi] \longrightarrow J^{\Delta_{\ell_1}}(\ell_2 N^+, p)(\mathbb{C})[\varpi] \\ & & \parallel \\ H^1(M^{\Delta_{\ell_1}}(N^+, p)(\mathbb{C}), \mathbb{F}(1)) \oplus H^1(M^{\Delta_{\ell_1}}(N^+, p)(\mathbb{C}), \mathbb{F}(1)) & \longrightarrow & H^1(M^{\Delta_{\ell_1}}(\ell_2 N^+, p)(\mathbb{C}), \mathbb{F}(1)) \\ \downarrow \simeq & & \downarrow \simeq \\ H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}(1)) \oplus H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}(1)) & \longrightarrow & H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(\ell_2 N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}(1)) \\ \downarrow \simeq & & \downarrow \simeq \\ H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}) \oplus H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}) & \xrightarrow{\alpha_{\ell_2}} & H_{\text{ét}}^1(M^{\Delta_{\ell_1}}(\ell_2 N^+, p)_{\overline{\mathbb{Q}}}, \mathbb{F}) \end{array}$$

We regard g_{ℓ_1} as a mod ϖ^n modular form on $M^{\Delta_{\ell_1}}(N^+, p)$ with trivial character. Let $\mathcal{I}_{g_{\ell_1}, p}^{\Delta_{\ell_1}} \subset \mathbb{T}^{\Delta_{\ell_1}}(N^+, p)$ be the ideal corresponding to g_{ℓ_1} and $\mathfrak{m}_{g_{\ell_1}, p}$ be the maximal ideal in $\mathbb{T}^{\Delta_{\ell_1}}(N^+, p)$. Since $\mathfrak{m}_{g_{\ell_1}, p}$ induces a residually irreducible Galois representation, we have $\text{Hom}(\Gamma_{\infty, N^+, p}, \mathbb{F}_p)[\mathfrak{m}_{g_{\ell_1}, p}] = 0$ due to Corollary 10.13. Therefore, $(\Gamma_{\infty, N^+, p})^{ab}/\mathfrak{m}_{g_{\ell_1}, p} = 0$ and also $(\Gamma_{\infty, N^+, p})^{ab}/\mathcal{I}_{g_{\ell_1}, p}^{\Delta_{\ell_1}} = 0$ due to Nakayama's lemma with $\mathbb{T}^{\Delta_{\ell_1}}(N^+, p)_{\mathfrak{m}_{g_{\ell_1}, p}}$. By Proposition 10.14, we have

$$\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})/\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})^{ss} \simeq (\Gamma_{\infty, N^+, p})^{ab}.$$

Thus, we have the following statement.

Lemma 10.21. *The image of $\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})^{ss}$ in $\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})/\mathcal{I}_{g_{\ell_1}, p}^{\Delta_{\ell_1}}$ is full.*

In order to reduce the level to the $\Gamma_0(N^+p)$ -structure, consider the natural map

$$\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2}) \rightarrow \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2}).$$

Its cokernel can be identified with the abelian quotient of the image of $\Gamma_0(p)$ in $\text{SL}_2(\mathbb{Z}/p\mathbb{Z})$, hence has order dividing $\varphi(p) = p - 1$, which is prime to p . Then the composition

$$\mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2}) \longrightarrow \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2}) \longrightarrow \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2})/\varpi^n$$

is surjective.

Remark 10.22. In the non-ordinary case, the level p is not in the level of the argument. Thus, we should replace N^+ by N^+/q and p by q under Condition Ihara (Assumption 2.8).

Since the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})^{ss} & \xrightarrow{\text{Lemma 10.21}} & \mathfrak{J}^{\Delta_{\ell_1}}(N^+, p)(\mathbb{F}_{\ell^2})/\mathcal{I}_{g_{\ell_1}, p}^{\Delta_{\ell_1}} \\ \downarrow & & \downarrow \text{the above surjection} \\ \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2})^{ss} & \longrightarrow & \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}, \end{array}$$

the map at the bottom

$$\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2})^{ss} \rightarrow \mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{\ell^2})/\mathcal{I}_{g_{\ell_1}}^{\Delta_{\ell_1}}$$

is also surjective.

Let $\bar{\rho}$ be the residual Galois representation associated to the maximal ideal $\mathfrak{m}_{g_{\ell_1\ell_2}} \subset \mathbb{T}^{\Delta_{\ell_1\ell_2}}(N^+p)$. Then $(\bar{\rho}, \Delta_{\ell_1\ell_2})$ satisfies Condition CR; thus, the following mod ϖ multiplicity one result holds as in [Kim17, Lemma 3.6] (also with the p -distinguished property of $\bar{\rho}$): $S_2^{\Delta_{\ell_1\ell_2}}(N^+p, \mathbb{F})_{\mathfrak{m}_{g_{\ell_1\ell_2}}}[\mathfrak{m}_{g_{\ell_1\ell_2}}] \simeq \mathbb{F}$. Then the argument of Wiles [Wil95, §2] shows that $S_2^{\Delta_{\ell_1\ell_2}}(N^+p, \mathcal{O})_{\mathfrak{m}_{g_{\ell_1\ell_2}}}$ is a free module over $\mathbb{T}_2^{\Delta_{\ell_1\ell_2}}(N^+p)_{\mathfrak{m}_{g_{\ell_1\ell_2}}}$ of rank one. This implies that $g_{\ell_1\ell_2}$ is uniquely determined up to \mathcal{O}_n^\times as in [CH15, Proof of Theorem 5.5].

Remark 10.23. In [CH15], such a multiplicity one result comes from the Taylor-Wiles system argument under Condition CR⁺. See [CH16, §6] for detail.

11. CONSTRUCTION OF COHOMOLOGY CLASSES ARISING FROM HEEGNER POINTS

We fix the following data (as input):

- an n -admissible form $\mathcal{D} = (\Delta, g)$;
- an n -admissible prime $\ell \nmid \Delta$.

11.1. Heegner points. For $m = \prod q_i^{r_i}$, write $m_0 = \prod q_i^{r_i+1}$. Consider the corresponding Shimura curve $M^{\Delta_\ell}(N^+)$. To each $a \in \widehat{K}^\times$, one can associate the Heegner point

$$P_{m_0}^{\Delta_\ell}(a) \in M^{\Delta_\ell}(N^+p)(H(m_0)) \cap \text{CM}_{K, \ell}(M^{\Delta_\ell}(N^+p))$$

using Shimura's reciprocity law.

Remark 11.1. For explicit definition of those points, see [CH15, (4.8)]. Since our Shimura curve has the $\Gamma_0(p)$ -level structure, we do not impose condition on r with $m = m'p^r$, which is given in [CH15, §4.3.2] and stated in [CH15, Theorem 5.1]. Their Shimura curves have level p^n with condition $r \geq n$ in order to use “mod ϖ^n Hida theory” for reduction of weight. See [CH15, Proposition 4.2] for their weight reduction argument.

11.2. Construction. Let $P_{m_0}^{\Delta\ell} := P_{m_0}^{\Delta\ell}(1) \in M^{\Delta\ell}(N^+p)$. Choose an auxiliary prime $q_0 \nmid p\ell N^+\Delta$ such that $1 + q_0 - a_{q_0}(f) \in \mathcal{O}^\times$. We define

$$\begin{aligned} \xi_{q_0}^{\Delta\ell} : \text{Div } M^{\Delta\ell}(N^+p)(H(m_0)) &\rightarrow J^{\Delta\ell}(N^+p)_{\mathcal{O}} \\ P &\mapsto \xi_{q_0}^{\Delta\ell}(P) := [(1 + q_0 - T_{q_0})P] \otimes (1 + q_0 - a_{q_0})^{-1}, \end{aligned}$$

and define

$$D_m^{\Delta\ell} := \sum_{\sigma \in \text{Gal}(H(m_0)/K(m))} \xi_{q_0}^{\Delta\ell} \left((P_{m_0}^{\Delta\ell})^\sigma \right) \in J^{\Delta\ell}(N^+p)(K(m))_{\mathcal{O}}.$$

Let

$$\text{Kum}^{\Delta\ell} : J^{\Delta\ell}(N^+p)(K(m))_{\mathcal{O}} \rightarrow H^1(K(m), \text{Ta}_p(J^{\Delta\ell}(N^+p))_{\mathcal{O}})$$

be the Kummer map. Define

$$\kappa_{\mathcal{D}}(\ell)_m := \kappa_{\mathcal{D}}(\ell)_{m'p^r} = \frac{1}{\alpha^r} \cdot \text{Kum}^{\Delta\ell}(D_m^{\Delta\ell}) \pmod{\mathcal{I}_{g_\ell}^{\Delta\ell}} \in H^1(K(m), \text{Ta}_p(J^{\Delta\ell}(N^+))_{\mathcal{O}}/\mathcal{I}_{g_\ell}^{\Delta\ell}) \simeq H^1(K(m), T_{f,n}).$$

Then these cohomology classes satisfy the norm compatibility relation

$$\text{cores}_{K(m'p^{r-1})}^{K(m'p^r)}(\kappa_{\mathcal{D}}(\ell)_{m'p^r}) = \kappa_{\mathcal{D}}(\ell)_{m'p^{r-1}}.$$

Thus, we have the norm compatible system

$$\kappa_{\mathcal{D}}(\ell)_{m'p^\infty} := (\kappa_{\mathcal{D}}(\ell)_{m'p^r})_r \in H^1(K(m'p^\infty), T_{f,n}).$$

The following proposition concerns the local properties of $\kappa_{\mathcal{D}}(\ell)_{m'p^\infty}$.

Proposition 11.2 ([CH15, Proposition 4.7]). $\kappa_{\mathcal{D}}(\ell)_{m'p^\infty} \in \text{Sel}_{\Delta\ell}(K(m'p^\infty), T_{f,n})$.

Proof. We need to check the following local statements:

- (1) $\partial_q(\kappa_{\mathcal{D}}(\ell)_m) = 0$ for $q \nmid p\Delta\ell m'$;
- (2) $\partial_q(\kappa_{\mathcal{D}}(\ell)_m) = 0$ for $q \mid N^+$;
- (3) $\partial_q(\kappa_{\mathcal{D}}(\ell)_m) = 0$ for $q \mid m'$;
- (4) $\text{res}_q(\kappa_{\mathcal{D}}(\ell)_m) \in H_{\text{ord}}^1(K(m)_q, T_{f,n})$ for $q \mid \Delta\ell$; and
- (5) $\text{res}_q(\kappa_{\mathcal{D}}(\ell)_m) \in H_{\text{ord}}^1(K(m)_q, T_{f,n})$ for $q \mid p$.

Part (1) comes from the fact that $J^{\Delta\ell}(N^+)$ has good reduction at primes not dividing $\ell\Delta N^+$ and $K(m)$ is unramified at primes not dividing m . Part (2) comes from Proposition 5.9.(1) for $q \mid N^+$. The elephantine condition is used here. Part (3) comes from the following argument.

Consider the following horizontally exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{ur}}^1(K(p^r)_q, T_{f,n}) & \xrightarrow{\text{inf}} & H^1(K(p^r)_q, T_{f,n}) & \xrightarrow{\text{res}} & H^1(I_{K(p^r)_q}, T_{f,n}) \\ & & \downarrow \text{res} \simeq & & \downarrow \text{res} \simeq & & \downarrow \text{res} \simeq \\ 0 & \longrightarrow & H_{\text{ur}}^1(K(m)_q, T_{f,n}) & \xrightarrow{\text{inf}} & H^1(K(m)_q, T_{f,n}) & \xrightarrow{\text{res}} & H^1(I_{K(m)_q}, T_{f,n}). \end{array}$$

Then the latter two vertical restriction maps are isomorphism because $K(m)/K(p^r)$ is an extension of degree prime-to- p and the former vertical isomorphism comes from these two isomorphisms. Since $K(p^r)$ is unramified at q , it reduces to Part (1).

Part (4) comes from the fact that $J^{\Delta\ell}(N^+)$ has purely toric reduction at primes dividing $\Delta\ell$. Part (5) comes from the exactly same argument given in [CH15, Proof of Proposition 4.7]. For a more detail, see [Lon12, Lemma 7.16] and [CH15, Proposition 4.7]. \square

Remark 11.3. As mentioned in [BD05, Remark 3 after Assumption 6, Introduction], our cohomology classes do not use any Kolyvagin derivative. Thus, we do not see any transverse local condition here.

11.3. Reduction to Gross points. Reminding the definition of red_V in §9.5 and the explicit definition of Gross and Heegner points as in [CH15, §4.3.1 and (4.8)], respectively, Heegner points map to Gross points following diagram (4) in §9.4.2.

$$\begin{array}{ccc} P_{m_0}(a) & \xrightarrow{cl} & cl(P_{m_0}(a)) \\ cl_V \downarrow & & \downarrow \text{red}_\ell \\ \psi_{m_0}(a) & \xrightarrow{\simeq} & \psi_{m_0}(a) \end{array}$$

where $\text{red}_V \circ \iota_\ell = cl_V$.

Remark 11.4. In some sense, the first explicit reciprocity law is a “deformation” of this reduction to group rings (and the Iwasawa algebra) following Theorem 10.9.(2). This reduction of Heegner points to Gross points is the Euler system interpretation of the original form of Jochnowitz congruences [BD99, Theorem 6.1]. The connection can be made via the Gross-Zagier formula and the Gross-Walspurger-Zhang formula.

12. EXPLICIT RECIPROCITY LAWS

We review two explicit reciprocity laws à la Bertolini-Darmon, which connect the cohomology classes arising from Heegner points and theta elements arising from Gross points.

12.1. n -admissible theta elements and the connection with theta elements. Let $\mathcal{D} = (\Delta, g)$ be an n -admissible form. Then we define an element in $\mathcal{O}_n[G_{m_0}]$ depending on \mathcal{D} and ψ by

$$\tilde{\theta}_{m_0}(\mathcal{D}) := \sum_a g(\psi_{m_0}(a)) [a]_{m_0} \in \mathcal{O}_n[G_{m_0}]$$

as in §3. Then we define the n -admissible theta element associated to \mathcal{D} and ψ by the image of

$$\frac{1}{\alpha^r} \cdot \theta_{m_0}(\mathcal{D})$$

in $\mathcal{O}_n[\Gamma_m]$. Define the n -admissible Bertolini-Darmon element associated to \mathcal{D} by

$$L_p(K(m), \mathcal{D}) := \theta_m(\mathcal{D}) \cdot \theta_m^*(\mathcal{D}) \in \mathcal{O}_n[\Gamma_m].$$

Proposition 12.1 (Existence of n -admissible forms; [CH15, Proposition 6.13]). *If $\Delta = N^-$, there exists an n -admissible form $\mathcal{D}_n^f = (N^-, f_{\alpha,n})$ such that*

$$\theta_m(\mathcal{D}_n^f) \equiv \theta_m(f_\alpha) \pmod{\varpi^n}$$

in $\mathcal{O}_n[\Gamma_m]$ up to a unit in \mathcal{O}_n .

Remark 12.2 (on the existence of n -admissible forms). All the n -admissible forms in this article come from the form \mathcal{D}_n^f and its level raisings at two n -admissible primes §10.3. Thus, the Euler system argument itself is non-empty.

12.2. The first explicit reciprocity law. Let ℓ be an n -admissible prime. The first explicit reciprocity law connects the cohomology classes arising from Heegner points on the Shimura curve of discriminant $\Delta\ell$ and Gross points on the Gross curve of discriminant Δ .

Theorem 12.3 (The first explicit reciprocity law; [BD05, Theorem 4.1], [CH15, Theorem 5.1]). *Assume Condition CR. If ℓ is an n -admissible prime, then*

$$v_\ell(\kappa_{\mathcal{D}}(\ell)_m) = 0$$

in $H_{\text{fin}}^1(K(m)_\ell, T_{f,n}) \simeq \mathcal{O}_n[\Gamma_m]$, and the equality

$$\partial_\ell(\kappa_{\mathcal{D}}(\ell)_m) = \theta_m(\mathcal{D})$$

holds under identifications $H_{\text{ord}}^1(K(m)_\ell, T_{f,n}) \simeq \mathcal{O}_n[\Gamma_m]$ up to multiplication by \mathcal{O}_n^\times and Γ_m .

Remark 12.4. The equality depends on the choices of the embedding ι_p and the isomorphism

$$\text{Ta}_p(J^{\Delta\ell}(N^+p))_{\mathcal{O}} / \mathcal{I}_g^{\Delta\ell} \simeq T_{f,n},$$

and different choices result in the “up to multiplication by \mathcal{O}_n^\times and Γ_m ”-part.

Proof. The reduction of Heegner points to Gross points in §11.3, the diagram in Theorem 10.9.(2), and the duality of the spaces of modular forms with ψ_g directly imply the conclusion. The explicit computation is given in [CH15, Proof of Theorem 5.1]. \square

12.3. The second explicit reciprocity law. Let ℓ_1 and ℓ_2 be n -admissible primes.

Theorem 12.5 (Second explicit reciprocity law). *Assume Condition CR and Condition Ihara. Then there exists an n -admissible form $\mathcal{D}^{\ell_1\ell_2} = (\Delta_{\ell_1\ell_2}, g_{\ell_1\ell_2})$ such that*

$$v_{\ell_1}(\kappa_{\mathcal{D}}(\ell_2))_m = v_{\ell_2}(\kappa_{\mathcal{D}}(\ell_1))_m = \theta_m(\mathcal{D}^{\ell_1\ell_2})$$

holds under the identifications $H_{\text{fin}}^1(K(m)_{\ell_1}, T_{f,n}) \simeq H_{\text{fin}}^1(K(m)_{\ell_2}, T_{f,n}) \simeq \mathcal{O}_n[\Gamma_m]$ up to multiplication by \mathcal{O}_n^\times and Γ_m .

Proof. Using the moduli interpretation of Shimura curves $M^{\Delta_{\ell_1\ell_2}}(N^+p)$ as in [CH15, §5.3], we can observe

$$P_{m_0}^{\Delta_{\ell_1}}(a) \pmod{(\ell_2)^2} = \psi_{m_0}(a)\tau$$

under the identification

$$\mathfrak{J}^{\Delta_{\ell_1}}(N^+p)(\mathbb{F}_{(\ell_2)^2})^{ss} \simeq \mathbb{Z}[B^{\Delta_{\ell_1\ell_2}, \times} \setminus \widehat{B}^{\Delta_{\ell_1\ell_2}, \times} / \widehat{R}_{N^+p}^\times].$$

Using this identity, the computation is straightforward. \square

13. THE SUPPLY OF n -ADMISSIBLE PRIMES AND n -ADMISSIBLE SETS

We focus on the infinitude of certain n -admissible primes with help of the Chebotarev density theorem. It is a generalization of [BD94, Lemma 2.21], [BD05, Theorem 3.2], and [CH15, Theorem 6.3] for our purpose, i.e. not over K but over $K(m')$. It is one of the key technical improvements to remove the difficulties in the generalization of the Euler system argument. If $m' = 1$, the statement becomes the original statement.

Theorem 13.1. *Let $n_0 \leq n$ be positive integers. Let m' be a positive integer with $(m', Np) = 1$. Let s be a non-zero element of $H^1(K(m'), A_{f,n_0})$. Assume that $A_{f,1}$ has big image (Assumption 1.15). Then there exist infinitely many n -admissible primes ℓ such that $\partial_\ell(s) = 0$ and the map*

$$v_\ell : \langle s \rangle \rightarrow H_{\text{fin}}^1(K(m')_\ell, A_{f,n_0})$$

is injective, where $\langle s \rangle$ is the \mathcal{O} -submodule of $H^1(K(m'), A_{f,n_0})$ generated by s .

Proof. Suppose that $\langle s \rangle \simeq \mathcal{O}/\varpi^r \mathcal{O}$ with $r < n_0$. Since the map

$$H^1(K(m')_\ell, A_{f,1}) \rightarrow H^1(K(m')_\ell, A_{f,n_0})$$

induced by the inclusion is injective for all n -admissible prime ℓ , we replace s by $\varpi^{r-1}s$. Thus, we may assume $s \in H^1(K(m')_\ell, A_{f,1})$. Let $\bar{\rho}$ be the residual representation, and $F := \overline{\mathbb{Q}}^{\ker(\bar{\rho})}$. Since $\bar{\rho}$ is unramified outside Np and $(\text{disc}(K)m', Np) = 1$, $K(m')$ and F are linearly disjoint. Let $M(m') = K(m')F$ and $M = KF$ be the compositum fields. Then we have

$$\begin{aligned} \text{Gal}(M(m')/\mathbb{Q}) &= \text{Gal}(K(m')/\mathbb{Q}) \times \text{Gal}(F/\mathbb{Q}) \\ \text{Gal}(M/\mathbb{Q}) &= \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(F/\mathbb{Q}) \end{aligned}$$

Since $\text{Gal}(K(m')/\mathbb{Q}) = \text{Gal}(K(m')/K) \rtimes \text{Gal}(K/\mathbb{Q})$, the element can be written as $(\gamma_{m'}^i, \tau^j)$ where τ is the complex conjugate, $i \in \{0, \dots, m' - 1\}$, and $j \in \{0, 1\}$.

We identify

$$\begin{aligned} \text{Gal}(M(m')/\mathbb{Q}) &= \text{Gal}(K(m')/\mathbb{Q}) \times \text{Gal}(F/\mathbb{Q}) \subseteq \text{Gal}(K(m')/\mathbb{Q}) \times \text{Aut}_{\mathbb{F}}(A_{f,1}) \\ \text{Gal}(M/\mathbb{Q}) &= \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(F/\mathbb{Q}) \subseteq \text{Gal}(K/\mathbb{Q}) \times \text{Aut}_{\mathbb{F}}(A_{f,1}) \end{aligned}$$

as subgroups, respectively.

Thus, we may write

- an element of $\text{Gal}(M(m')/\mathbb{Q})$ as a triple $(\gamma_{m'}^i, \tau^j, \sigma)$ with $\sigma \in \text{Aut}_{\mathbb{F}}(A_{f,1})$ and
- an element of $\text{Gal}(M/\mathbb{Q})$ as a pair (τ^j, σ) with $\sigma \in \text{Aut}_{\mathbb{F}}(A_{f,1})$.

By Assumption 1.15 (and [CH15, Lemma 6.1]), $\text{Gal}(F/\mathbb{Q}) \simeq \overline{\rho}(G_{\mathbb{Q}})$ contains $-I_2 \in \text{GL}_2(\mathbb{F})$.

Since $\text{Gal}(M(m')/K(m')) \simeq \text{Gal}(F/\mathbb{Q})$, the inflation-restriction sequence shows that

$$\begin{array}{ccc} H^1(\text{Gal}(M(m')/K(m'))/\{\pm I_2\}, A_{f,1}^{\{\pm I_2\}}) & \longrightarrow & H^1(\text{Gal}(M(m')/K(m')), A_{f,1}) \\ \uparrow & & \downarrow \\ 0 & & H^1(\{\pm I_2\}, A_{f,1})^{\text{Gal}(M(m')/K(m'))/\{\pm I_2\}} \end{array}$$

is exact. Since $-I_2$ does not fix any non-zero vector in $A_{f,1}$ and $\{\pm I_2\}$ is of order 2, we have

$$H^1(\text{Gal}(M(m')/K(m')), A_{f,1}) = 0.$$

Therefore, the restriction map

$$\begin{array}{ccc} H^1(K(m'), A_{f,1}) & \longrightarrow & H^1(M(m'), A_{f,1}) = \text{Hom}(G_{M(m')}, A_{f,1}) \\ s \longmapsto & & \overline{s} \end{array}$$

is injective. Let $M(m')_s$ be the non-trivial extension of M cut out by the image \overline{s} of s under the restriction to $H^1(M(m'), A_{f,1}) = \text{Hom}(G_{M(m')}, A_{f,1})$; thus, $M(m')_s/M(m')$ is an abelian extension. Let

$$A_s := \text{Gal}(M(m')_s/M(m')) \subseteq A_{f,1}$$

under \overline{s} .

Without loss of generality, we assume that s belongs to a τ -isotypic subspace such that $\tau s = \delta s$ with $\delta = \pm 1$. Then, under the identification $A_s \subseteq A_{f,1}$, A_s admits the action of $\sigma \in \text{Gal}(F/\mathbb{Q})$ via $\overline{\rho}$ and $\tau \in \text{Gal}(K/\mathbb{Q})$ by the formula

$$(\tau^j, \sigma)(v) = \delta^j \overline{\rho}(\sigma)v$$

where $v \in A_s$. However, A_s itself has no reason to be stable under the action of $\gamma_{m'}^i \in \text{Gal}(K(m')/K)$. Thus, $M(m')_s/\mathbb{Q}$ may not be Galois.

We explicitly construct the Galois closure L of $M(m')_s/\mathbb{Q}$. We also show that

- (1) $A_L := \text{Gal}(L/M(m'))$ is an abelian extension and
- (2) $\text{Gal}(L/\mathbb{Q})$ can be written as $\text{Gal}(L/M(m')) \rtimes \text{Gal}(M(m')/\mathbb{Q})$.

Since $H^1(K(m'), A_{f,1})$ admits the action of $\Gamma_{m'} = \text{Gal}(K(m')/K)$, we may consider the $\Gamma_{m'}$ -orbit of s

$$\Gamma_{m'} s = \{\gamma_{m'}^i s : i = 0, \dots, m' - 1\}.$$

Consider the extension $M(m')_{\gamma_{m'}^i s}$ of $\gamma_{m'}^{-i} M(m') = M(m')$ using $\overline{\gamma_{m'}^i s}$ with

$$A_{\gamma_{m'}^i s} := \text{Gal}(M(m')_{\gamma_{m'}^i s}/M(m')) \subseteq A_{f,1}$$

under $\overline{\gamma_{m'}^i s}$. Then $A_{\gamma_{m'}^i s} \subseteq A_{f,1}$ also admits the action of $\text{Gal}(F/\mathbb{Q})$ via $\overline{\rho}$ as in the case of A_s since the actions of $\text{Gal}(F/\mathbb{Q})$ and $\text{Gal}(K(m')/\mathbb{Q})$ commute with each other.

Consider the set of all such Galois groups over $M(m')$

$$\{A_{\gamma_{m'}^i s} = \text{Gal}(M(m')_{\gamma_{m'}^i s}/M(m')) : i = 0, \dots, m' - 1\}.$$

Then this set admits the action of $\Gamma_{m'}$ and τ by

$$\begin{aligned} \gamma_{m'} \left(A_{\gamma_{m'}^i s} \right) &= A_{\gamma_{m'}^{i+1} s} \\ \tau \left(A_{\gamma_{m'}^i s} \right) &= A_{\gamma_{m'}^{-i} s}. \end{aligned}$$

We define L by the compositum of $M(m')_{\gamma_{m'}^i s}$ for all $i = 0, \dots, m' - 1$ and then $A_L := \text{Gal}(L/M(m'))$ is the subgroup of $A_{f,1}$ generated by $A_{\gamma_{m'}^i s}$ for all $i = 0, \dots, m' - 1$. Then L is the Galois closure of $M(m')_s/\mathbb{Q}$ by construction, and A_L is abelian since $A_L \subset A_{f,1}$. Also, A_L admits the action of $\text{Gal}(K(m')/K)$ by construction as well as the action of $\text{Gal}(F/\mathbb{Q})$ via $\overline{\rho}$ and $\text{Gal}(K/\mathbb{Q})$ via τ . Thus, L is Galois over \mathbb{Q} and $\text{Gal}(L/\mathbb{Q}) = A_L \rtimes \text{Gal}(M(m')/\mathbb{Q})$.

Each element of $\text{Gal}(L/\mathbb{Q})$ can be written as a quadruple $(v', \gamma_{m'}^i, \tau^j, \sigma)$ where $v' \in A_L$, $\gamma_{m'}^i \in \Gamma_{m'}$, $\tau^j \in \text{Gal}(K/\mathbb{Q})$, and $\sigma \in \text{Gal}(F/\mathbb{Q})$.

Due to Assumption 1.15, there exists an element $h \in \overline{\rho}(G_{\mathbb{Q}})$ such that $\text{tr}(h) = \det(h) + 1$ with $\det(h) \neq \pm 1 \in \mathbb{F}$ ([CH15, Lemma 6.2]). It implies that we can choose a quadruple $(v', 1, \tau, \sigma)$ as an element of $\text{Gal}(L/\mathbb{Q})$ such that:

- (1) $\overline{\rho}(\sigma) \in \text{GL}_2(\mathbb{F})$ has eigenvalue $\delta (= \pm 1)$ and $\lambda \in \mathbb{F}^{\times}$ where $\lambda \not\equiv \pm 1 \pmod{\varpi}$ and the order of λ is prime to p ;
- (2) the element $v' \in A_L$ satisfies that its image v in $\text{Gal}(M(m')_s/M(m'))$ is non-zero, and v belongs to the δ -eigenspace for σ under $\overline{\rho} = \rho_1$.

Note that $\text{Gal}(L/M(m')_s)$ is normal in $\text{Gal}(L/M(m'))$ since $\text{Gal}(M(m')_s/M(m'))$ is Galois.

By the Chebotarev density theorem, there exist infinitely many primes ℓ with $\ell \nmid Nm'p$ such that ℓ is unramified in L/\mathbb{Q} and satisfies $\text{Frob}_{\ell}(L/\mathbb{Q})$ is the quadruple we have chosen; in fact, the set of such primes has positive density. Then the restriction to $M(m')/\mathbb{Q}$ becomes the triple $(1, \tau, \sigma)$. It implies that ℓ is n -admissible.

For each prime ℓ of $M(m')$ above ℓ , let d be the residual degree of ℓ/ℓ , which is even. Then we have

$$\begin{aligned} \text{Frob}_{\ell}(L/M(m')) &= \text{Frob}_{\ell}(L/\mathbb{Q})^d \\ &= (v', 1, \tau, \sigma)^d \\ &= v' + \delta \sigma v' + \sigma^2 v' + \cdots + \delta \sigma^{d-1} v'. \end{aligned}$$

Since the order of σ is prime to p due to property (1) of the quadruple, d is prime to p .

We regard $\overline{s} \in \text{Hom}(G_M, A_{f,1})$ as the natural quotient map

$$\overline{s} : \text{Gal}(L/M(m')) \simeq A_L \rightarrow \text{Gal}(M(m')_s/M(m')) \simeq A_s.$$

It is equivariant under the action of τ and $\sigma \in \text{Gal}(F/\mathbb{Q})$. Note that A_s is stable under those actions. Then we have

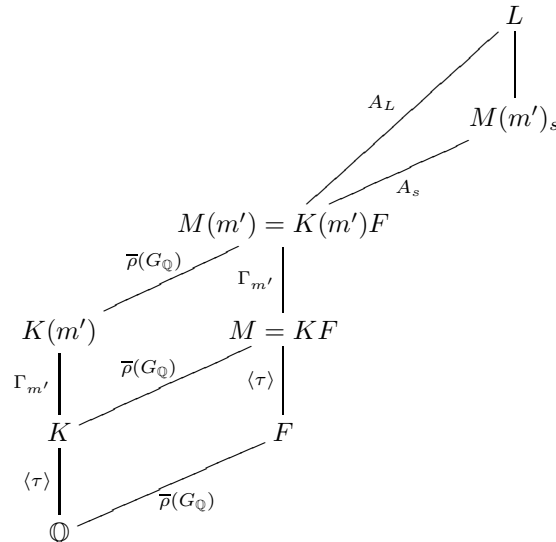
$$\begin{aligned} \overline{s}(\text{Frob}_{\ell}(L/M(m'))) &= \overline{s}(v' + \delta \sigma v' + \sigma^2 v' + \cdots + \delta \sigma^{d-1} v') \\ &= \overline{s}(v') + \delta \sigma \overline{s}(v') + \sigma^2 \overline{s}(v') + \cdots + \delta \sigma^{d-1} \overline{s}(v') \\ &= v + \delta \sigma v + \sigma^2 v + \cdots + \delta \sigma^{d-1} v \\ &= d \cdot v \\ &\neq 0 \end{aligned}$$

due to property (2) of the quadruple. Thus, we have

$$v_{\ell}(s) = \overline{s}(\text{Frob}_{\ell}(L/M(m')))|_{\ell} \neq 0.$$

□

The following diagram would be helpful to understand the proof of Theorem 13.1:



Let Δ be a square-free integer such that Δ/N^- is a product of n -admissible primes.

Definition 13.2 (n -admissible set). A finite set \mathfrak{S} of primes is said to be an n -admissible set for f and m' if

- (1) All $\ell \in \mathfrak{S}$ are n -admissible for f
- (2) The map $\text{Sel}_\Delta(K(m'), A_{f,n}) \rightarrow \bigoplus_{\ell \in \mathfrak{S}} H_{\text{fin}}^1(K(m')_\ell, A_{f,n})$ is injective.

For a given finite set of n -admissible primes satisfying the statement in Theorem 13.1, we are always able to enlarge the set to an n -admissible set for f and m' . This is one of the standard applications of Theorem 13.1.

Proposition 13.3. *Any finite collection of n -admissible primes can be enlarged to an n -admissible set for f and m' .*

14. CONTROL THEOREMS AND THE FREENESS OF COMPACT SELMER GROUPS

We prove the freeness theorem of Δ -ordinary S -relaxed Selmer groups with n -admissible set \mathfrak{S} where $S = \prod_{q \in \mathfrak{S}} q$. The original argument depends heavily on the properties of p -extensions. In order to generalize to the general cyclic case, we use the decomposition of modules over group rings.

14.1. Controlling discrete Selmer groups. Let L/K be a subextension of $K(m'p^\infty)$. Let S be a square-free product of n -admissible primes.

Lemma 14.1. *Assume that the residual representation $A_{f,1}$ of $G_\mathbb{Q}$ is absolutely irreducible. Then the restriction map*

$$H^1(K, A_{f,n}) \rightarrow H^1(L, A_{f,n})^{\text{Gal}(L/K)}$$

is an isomorphism.

Proof. Since $K(m'p^r)$ over \mathbb{Q} is Galois for all r , the residual irreducibility of $\bar{\rho}$ implies that $(T_{f,n})^{G_{K(m'p^r)}} = 0$. Then the isomorphism follows from the inflation-restriction sequence. For any subextension of $K(m'^r)$, the proof is identical. \square

Lemma 14.2 (Exact control theorem). *Assume that*

- *the residual representation $A_{f,1}$ of $G_\mathbb{Q}$ is absolutely irreducible, and*
- *f satisfies Condition PO.*
- *f is N^+ -minimal.*

Then the restriction map

$$\text{Sel}_\Delta^S(K, A_{f,n}) \rightarrow \text{Sel}_\Delta^S(L, A_{f,n})^{\text{Gal}(L/K)}$$

is an isomorphism.

Proof. By Lemma 14.1, we know the map $\text{Sel}_\Delta^S(K, A_{f,n}) \rightarrow \text{Sel}_\Delta^S(L, A_{f,n})^{\text{Gal}(L/K)}$ is injective. To check the surjectivity, we check the following local statements:

- (1) $H^1(K_\ell, A_{f,n}/F_\ell^+ A_{f,n}) \rightarrow H^1(L_\ell, A_{f,n}/F_\ell^+ A_{f,n})$ is injective for $\ell \mid p\Delta$; and
- (2) $H^1(K_\ell^{\text{ur}}, A_{f,n}) \rightarrow H^1(L_\ell^{\text{ur}}, A_{f,n})$ is injective for $\ell \nmid p\Delta$.

Part (1) for $\ell \mid \Delta$ follows from $K_\lambda = L_\lambda$ for any prime λ of L above ℓ which is inert in K/\mathbb{Q} . Part (1) for $\ell = p$ comes from Lemma 5.12.

Since L/K is unramified outside $m = m'p^r$ and anticyclotomic, we have $K_\lambda^{\text{ur}} = L_\lambda^{\text{ur}}$ for each place λ of L above ℓ not dividing pm' .

For a place λ of L dividing m' , the kernel of the map is $H^1(L_\lambda^{\text{ur}}/K_\lambda^{\text{ur}}, (A_{f,n})^{G_{L_\lambda^{\text{ur}}}})$. Since we have $L_\lambda^{\text{ur}} = (L \cap K(\ell^r))_\lambda^{\text{ur}}$ where λ divides a prime ℓ with $\ell^r \parallel m'$, $\text{Gal}(L_\lambda^{\text{ur}}/K_\lambda^{\text{ur}})$ is an ℓ -group, which is prime to p . By [Ser95, Corollary 1, Page 130] and the identification with Tate cohomology, the kernel vanishes. \square

Lemma 14.3. *Assume that the residual representation $A_{f,1}$ of $G_\mathbb{Q}$ is absolutely irreducible. Then*

$$H^1(L, A_{f,n}) = H^1(L, A_f)[\varpi^n].$$

Proof. See [CH15, Proposition 1.9]. \square

Lemma 14.4 (Exact control theorem of reduction mod ϖ^n). *Assume that*

- *the residual representation $A_{f,1}$ of $G_\mathbb{Q}$ is absolutely irreducible,*
- *$(\bar{\rho}, \Delta)$ satisfies Condition CR,*

- f satisfies Condition PO, and
- f is N^+ -minimal.

Then

$$\mathrm{Sel}_\Delta^S(L, A_{f,n}) = \mathrm{Sel}_\Delta^S(L, A_f)[\varpi^n].$$

Proof. See [CH15, Proposition 1.9]. \square

The following corollary is the combination of Lemma 14.1 and Lemma 14.3.

Corollary 14.5. *Assume that the residual representation $A_{f,1}$ of $G_{\mathbb{Q}}$ is absolutely irreducible. Let $\mathfrak{m}_{L,n} = (\varpi, \gamma_L - 1)$ be a maximal ideal of $\mathcal{O}_n[[\mathrm{Gal}(L/K)]]$ where γ_L is a generator of $\mathrm{Gal}(L/K)$. Then the restriction map*

$$H^1(K, A_{f,1}) \rightarrow H^1(L, A_{f,n})[\mathfrak{m}_{L,n}]$$

is an isomorphism.

14.2. The stability of n -admissible sets and the Poitou-Tate sequence. In this subsection, the assumptions of Lemma 14.2 are in force.

Let L/K be a subextension of $K(m)$ and \mathfrak{S} an n -admissible set for f and m' . Let $L(m') := L \cap K(m')$.

The following lemma shows that the concept of n -admissible sets is stable under base change of p -power degree.

Lemma 14.6. *The natural map*

$$\mathrm{Sel}_\Delta(L, A_{f,n}) \rightarrow \bigoplus_{\ell \in \mathfrak{S}} H_{\mathrm{fin}}^1(L_\ell, A_{f,n})$$

is injective.

Proof. Let C be the kernel of the map. If C is non-zero, then there exists a non-trivial element s in C fixed by $\mathrm{Gal}(L/L(m'))$ since $\mathrm{Gal}(L/L(m'))$ is a p -group with help of the orbit-stabilizer theorem. Then s belongs to $\mathrm{Sel}_\Delta(L, A_{f,n})^{\mathrm{Gal}(L/L(m'))} = \mathrm{Sel}_\Delta(L(m'), A_{f,n})$ due to Lemma 14.2 with $S = 1$. Then it contradicts the definition of n -admissible sets for f and m' . \square

Let $S = \prod_{q \in \mathfrak{S}} q$.

Lemma 14.7 (Poitou-Tate exact sequences for Selmer groups). *We have an exact sequence*

$$0 \rightarrow \mathrm{Sel}_\Delta(L, T_{f,n}) \rightarrow \mathrm{Sel}_\Delta^S(L, T_{f,n}) \rightarrow \bigoplus_{\ell \in \mathfrak{S}} H_{\mathrm{sing}}^1(L_\ell, T_{f,n}) \rightarrow \mathrm{Sel}_\Delta(L, A_{f,n})^\vee \rightarrow 0$$

where $\mathrm{Sel}_\Delta(L, A_{f,n})^\vee = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Sel}_\Delta(L, A_{f,n}), E/\mathcal{O})$.

Proof. The Poitou-Tate sequences for Δ -ordinary Selmer groups and Lemma 14.6 directly imply the conclusion. See [CH15, Lemma 6.7] for detail. \square

14.3. The freeness of compact Selmer groups. In this subsection, we generalize [CH15, Proposition 6.8], which is based on [BD94, Theorem 3.2], to the general cyclic case in detail. The key idea is to apply the enhanced isotypic decomposition in §4.3 for the reduction to the p -power case. The assumptions of Lemma 14.2 are in force as before.

14.3.1. Statement. Let L/K be a subextension of $K(m)$ and F/K be a subextension of K in L . Let \mathfrak{S} be an n -admissible set for f and m' . Let $S = \prod_{q \in \mathfrak{S}} q$.

Theorem 14.8. *The Selmer group $\mathrm{Sel}_\Delta^S(K(m), T_{f,n})$ is free of rank $\#\mathfrak{S}$ over $\mathcal{O}_n[\Gamma_m]$ where $\Gamma_m = \mathrm{Gal}(K(m)/K)$.*

Due to Lemma 4.10, it suffices to prove the following statement.

Proposition 14.9. *Let $\omega : \mathrm{Gal}(K(m')/K) \rightarrow \mathcal{O}_\omega$ be a character. Then the Selmer group*

$$\mathrm{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega$$

is free of rank $\#\mathfrak{S}$ over $\mathcal{O}_{\omega,n}[\Gamma_{p^r}]$.

We prove Proposition 14.9 in the next subsections and describe the consequences in the the last subsection.

14.3.2. *Step 1: Proof of the case for $r = 1$ and $n = 1$.* Let $\Gamma_p = \text{Gal}(K(m'p)/K(m'))$. Since Γ_p trivially acts on \mathcal{O}_ω , we have

$$\left(\text{Sel}_\Delta^S(K(m'p), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right)^{\Gamma_p} = \text{Sel}_\Delta^S(K(m'p), T_{f,1})^{\Gamma_p} \otimes_\omega \mathcal{O}_\omega.$$

By the control theorem (Lemma 14.2 with S), it coincides with

$$\text{Sel}_\Delta^S(K(m'), T_{f,1}) \otimes_\omega \mathcal{O}_\omega.$$

Due to the Poitou-Tate sequence (Lemma 14.7 with $L = K(m')$),

$$\text{Sel}_\Delta^S(K(m'), T_{f,1})$$

and

$$\bigoplus_{\ell \in \mathfrak{S}} H_{\text{sing}}^1(K(m')_\ell, T_{f,1})$$

have the same cardinality.

From now on, we focus on the vector space structure on them after tensoring \mathcal{O}_ω via ω . Due to Proposition 5.9.(4), we have

$$\bigoplus_{\ell \in \mathfrak{S}} H_{\text{sing}}^1(K(m')_\ell, T_{f,1}) \simeq \bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_{m'}].$$

Consider the following vertically exact commutative digram induced from the Poitou-Tate sequence (Lemma 14.7), Proposition 5.9.(4), and the enhanced isotypic decomposition (Lemma 4.10):

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow (?) \\
\text{Sel}_\Delta(K(m'), T_{f,n}) & \xrightarrow{\simeq} & \bigoplus_\omega \text{Sel}_\Delta(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega & \xrightarrow{\text{proj}_\omega} & \text{Sel}_\Delta(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \\
\downarrow & & \downarrow & & \downarrow \\
\text{Sel}_\Delta^S(K(m'), T_{f,n}) & \xrightarrow{\simeq} & \bigoplus_\omega \text{Sel}_\Delta^S(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega & \xrightarrow{\text{proj}_\omega} & \text{Sel}_\Delta^S(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{\ell \in \mathfrak{S}} H_{\text{sing}}^1(K(m')_\ell, T_{f,n}) & \xrightarrow{\simeq} & \bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_{m'}] & \xrightarrow{\simeq} & \bigoplus_\omega \bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_{m'}] \otimes_\omega \mathcal{O}_\omega & \xrightarrow{\text{proj}_\omega} & \bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_{m'}] \otimes_\omega \mathcal{O}_\omega \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Sel}_\Delta(K(m'), A_{f,n})^\vee & \xrightarrow{\simeq} & \bigoplus_\omega \text{Sel}_\Delta(K(m'), A_{f,n})^\vee \otimes_\omega \mathcal{O}_\omega & \xrightarrow{\text{proj}_\omega} & \text{Sel}_\Delta(K(m'), A_{f,n})^\vee \otimes_\omega \mathcal{O}_\omega \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where proj_ω is the projection to the ω -component. By the cardinality consideration,

$$\text{Sel}_\Delta(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \rightarrow \text{Sel}_\Delta^S(K(m'), T_{f,n}) \otimes_\omega \mathcal{O}_\omega$$

is injective (for any ω), so the (?)-part in the diagram becomes completed. Thus, we obtain

$$\begin{aligned}
\dim_{\mathbb{F}_\omega} \left(\text{Sel}_\Delta^S(K(m'), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) &= \dim_{\mathbb{F}_\omega} \left(\bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_{m'}] \otimes_\omega \mathcal{O}_\omega \right) \\
&= \#\mathfrak{S}
\end{aligned}$$

where \mathbb{F}_ω is the residue field of \mathcal{O}_ω .

The group ring $\mathbb{F}_\omega[\Gamma_p]$ is isomorphic to the local ring $\mathbb{F}_\omega[\varepsilon]/(\varepsilon^p)$ via map $\gamma_p \mapsto \varepsilon + 1$ where γ_p is a generator of Γ_p .

Every finitely generated $\mathbb{F}_\omega[\Gamma_p]$ -module M can be written as a direct sum of cyclic $\mathbb{F}_\omega[\Gamma_p]$ -modules and the number of summands is $\dim_{\mathbb{F}_\omega}(M^{\Gamma_p})$. Hence, we can write

$$\text{Sel}_\Delta^S(K(m'p), T_{f,1})^{\Gamma_p} \otimes_\omega \mathcal{O}_\omega = V_1 \oplus \cdots \oplus V_{\#\mathfrak{S}}$$

where each V_i is a cyclic module over $\mathbb{F}_\omega[\Gamma_p]$. Let $n_i = \dim_{\mathbb{F}_\omega}(V_i)$.

Applying the same argument with $L = K(m'p)$, we have

$$p \cdot \#\mathfrak{S} = \dim_{\mathbb{F}_\omega} \left(\bigoplus_{\ell \in \mathfrak{S}} H_{\text{sing}}^1(K(m'p)_\ell, T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = \dim_{\mathbb{F}_\omega} \left(\text{Sel}_\Delta^S(K(m'p), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = \sum_{i=1}^{\#\mathfrak{S}} n_i.$$

Since $n_i \leq p$ for all i , we have $n_i = p$ for all i . It immediately shows that $\text{Sel}_\Delta^S(K(m'p), T_{f,1}) \otimes_\omega \mathcal{O}_\omega$ is free of rank $\#\mathfrak{S}$ over $\mathbb{F}_\omega[\Gamma_p]$.

14.3.3. *Step 2: Proof of the case for arbitrary r and $n = 1$.* By the argument of Step 1, we still have

$$\dim_{\mathbb{F}_\omega} \left(\text{Sel}_\Delta^S(K(m'p^r), T_{f,1})^{\Gamma_{p^r}} \otimes_\omega \mathcal{O}_\omega \right) = \dim_{\mathbb{F}_\omega} \left(\text{Sel}_\Delta^S(K(m'), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = \#\mathfrak{S}.$$

Thus, it suffices to check that $\text{Sel}_\Delta^S(K(m'p^r), T_{f,1}) \otimes_\omega \mathcal{O}_\omega$ is a free $\mathbb{F}[\Gamma_{p^r}]$ -module. By [AW10, Theorem 6, Page 112], it suffices to prove

$$H_{\text{Tate}}^0 \left(\Gamma_{p^r}, \text{Sel}_\Delta^S(K(m'p^r), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = 0$$

where $H_{\text{Tate}}^q(G, M)$ is the Tate cohomology group.

We use the induction on r . Consider a subextension $K(m'p^{r-1})/K(m')$ with

$$\Gamma_p = \text{Gal}(K(m'p^r)/K(m'p^{r-1})) \simeq \mathbb{Z}/p\mathbb{Z}.$$

By Lemma 14.6, the set of places of $K(m'p^{r-1})$ lying above the places of $K(m')$ in \mathfrak{S} is also an n -admissible set for f and $m'p^{r-1}$. Thus, the same argument used in Step 1 shows that

$$H_{\text{Tate}}^0 \left(\Gamma_p, \text{Sel}_\Delta^S(K(m'p^r), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = 0.$$

Here we have that $\text{Sel}_\Delta^S(K(m'p^r), T_{f,1}) \otimes_\omega \mathcal{O}_\omega$ is free of rank $p^{r-1} \cdot \#\mathfrak{S}$ over $\mathbb{F}_\omega[\text{Gal}(K(m'p^r)/K(m'p^{r-1}))]$. By the induction hypothesis, we have

$$H_{\text{Tate}}^0 \left(\text{Gal}(K(m'p^{r-1})/K(m')), \text{Sel}_\Delta^S(K(m'p^{r-1}), T_{f,1}) \otimes_\omega \mathcal{O}_\omega \right) = 0.$$

Since the norm map is transitive, it completes a proof of Step 2.

14.3.4. *Step 3: Proof of the general case.* The following three lemmas immediately imply the general case.

Lemma 14.10 ([BD94, Lemma 3.3]). *Let $\left(\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \right) [\varpi_\omega]$ be the ϖ_ω -torsion submodule of $\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega$. Then*

$$\left(\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \right) [\varpi_\omega] \simeq (\mathbb{F}[\Gamma_m] \otimes_\omega \mathcal{O}_\omega)^{\#\mathfrak{S}} \simeq (\mathbb{F}_\omega[\Gamma_{p^r}])^{\#\mathfrak{S}}.$$

Proof. Via Lemma 14.4, we have $\text{Sel}_\Delta^S(K(m), T_{f,n})[\varpi] = \text{Sel}_\Delta^S(K(m), T_{f,1})$. Using the enhanced isotypic decomposition, we have

$$\begin{aligned} \text{Sel}_\Delta^S(K(m), T_{f,n})[\varpi] &= \left(\bigoplus_\omega \left(\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \right) \right) [\varpi] \\ &= \bigoplus_\omega \left(\left(\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \right) [\varpi_\omega] \right), \end{aligned}$$

and

$$\text{Sel}_\Delta^S(K(m), T_{f,1}) = \bigoplus_\omega \text{Sel}_\Delta^S(K(m), T_{f,1}) \otimes_\omega \mathcal{O}_\omega.$$

Thus, we have

$$\text{Sel}_\Delta^S(K(m), T_{f,1}) \otimes_\omega \mathcal{O}_\omega = \left(\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega \right) [\varpi_\omega].$$

Applying Step 2 to $\text{Sel}_\Delta^S(K(m), T_{f,1}) \otimes_\omega \mathcal{O}_\omega$, we obtain the conclusion. \square

Lemma 14.11 ([BD94, Lemma 3.4]). *The module $\text{Sel}_\Delta^S(K(m), T_{f,n}) \otimes_\omega \mathcal{O}_\omega$ is a free module over $\mathcal{O}_{\omega,n} = \mathcal{O}_\omega/\varpi_\omega^n \mathcal{O}_\omega$.*

Proof. Let $m = [K(m) : K]$. By Lemma 14.10, we have

$$\mathrm{Sel}_{\Delta}^S(K(m), T_{f,n}) \otimes_{\omega} \mathcal{O}_{\omega} \simeq \bigoplus_{i=1}^{p^r \cdot \#\mathfrak{S}} \mathcal{O}_{\omega, n_i}$$

with $n_i \leq n$. By Lemma 14.7 and the argument of Step 1, we have

$$\# \left(\mathrm{Sel}_{\Delta}^S(K(m), T_{f,n}) \otimes_{\omega} \mathcal{O}_{\omega} \right) = \# \left(\bigoplus_{\ell \in \mathfrak{S}} \mathbb{F}[\Gamma_m] \otimes_{\omega} \mathcal{O}_{\omega} \right) = \mathrm{Nm}(\varpi_{\omega})^{p^r \cdot n \cdot \#\mathfrak{S}}.$$

Thus, we have $n_i = n$ for all i . □

Lemma 14.12 ([BD94, Lemma 3.5]). *Let M be an $\mathcal{O}_n[\Gamma_{p^r}]$ -module such that*

- (1) $M[\varpi] \simeq (\mathbb{F}[\Gamma_{p^r}])^{\#\mathfrak{S}}$, and
- (2) M is \mathcal{O}_n -free.

Then $M \simeq (\mathcal{O}_n[\Gamma_{p^r}])^{\#\mathfrak{S}}$.

14.3.5. *Consequences.*

Corollary 14.13. *Under the same assumptions of Theorem 14.8, the natural map*

$$\mathrm{Sel}_{\Delta}^S(K(m), T_{f,n}) \rightarrow \mathrm{Sel}_{\Delta}^S(K(m''), T_{f,n})$$

with $m''|m$ is surjective.

Proof. This is a direct consequence of the definition of the corestriction map, Theorem 14.8, and the cardinality consideration. □

Taking projective limit with respect to r , we also have the following statement.

Corollary 14.14. *Under the same assumptions of Theorem 14.8, $\mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n})$ is free of rank $\#\mathfrak{S}$ over $\mathcal{O}_n[[\Gamma_{m'p^{\infty}}]]$.*

Corollary 14.15 (Control theorem with respect to quotients). *Under the same assumptions of Theorem 14.8, we have isomorphisms*

$$\begin{aligned} \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) / \mathfrak{m}_{p^{\infty},n} &\simeq \mathrm{Sel}_{\Delta}^S(K(m'), T_{f,1}) \\ \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) / \mathfrak{m}_{m'p^{\infty},n} &\simeq \mathrm{Sel}_{\Delta}^S(K, T_{f,1}) \\ \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) / \varpi &\simeq \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,1}) \end{aligned}$$

where $\mathfrak{m}_{p^{\infty},n} = (\gamma_{p^{\infty}} - 1, \varpi) \subset \mathcal{O}_n[[\Gamma_{m'p^{\infty}}]]$ and $\mathfrak{m}_{m'p^{\infty},n} = (\gamma_{m'p^{\infty}} - 1, \varpi) \subset \mathcal{O}_n[[\Gamma_{m'p^{\infty}}]]$.

Proof. This is a direct consequence of Theorem 14.8 and Corollary 14.13. □

15. THE EULER SYSTEM ARGUMENT: CHARACTER BY CHARACTER INDUCTION

This section is devoted to proving the character-evaluated version of the main theorem (Theorem 15.12) under the assumptions given in the main theorem.

15.1. Specialization via characters. We recall some material of §4.2 for φ . Let $\varphi : \mathcal{O}[[\Gamma_{m'p^{\infty}}]] \rightarrow \mathcal{O}_{\varphi}$ be an \mathcal{O} -algebra homomorphism where \mathcal{O}_{φ} is a discrete valuation ring of characteristic zero (depending on φ). Let ϖ_{φ} be a uniformizer of \mathcal{O}_{φ} and \mathfrak{m}_{φ} be the maximal ideal of \mathcal{O}_{φ} . Let $\mathbb{F}_{\varphi} := \mathcal{O}_{\varphi} / \varpi_{\varphi}$ and $\mathcal{O}_{\varphi,n} := \mathcal{O}_{\varphi} / \varpi_{\varphi}^n$. Let M be a finitely generated \mathcal{O}_{φ} -module. For $x \in M$, we define

$$\mathrm{ord}_{\varphi}(x) := \sup \{ t \in \mathbb{Z}_{\geq 0} : x \in \varpi_{\varphi}^t M \}.$$

Then $x = 0$ if and only if $\mathrm{ord}_{\varphi}(x) = \infty$.

For a $\mathcal{O}[[\Gamma_{m'p^{\infty}}]]$ -module M , $M \otimes_{\mathcal{O}} \mathcal{O}_{\varphi} := M \otimes_{\mathcal{O}[[\Gamma_{m'p^{\infty}}]]} \mathcal{O}_{\varphi}$ is called the **specialization of M via φ** , which is an \mathcal{O}_{φ} -module.

15.2. Setup for induction. Let n be a positive integer and $\Delta > 1$ be a square-free product of an odd number of primes satisfying

- $N^-|\Delta$, and
- Δ/N^- is a product of n -admissible primes.

For each n -admissible form $\mathcal{D} = (\Delta, g_n)$ as in Definition 10.1, we define a non-negative integer

$$t_{\mathcal{D}} := \text{ord}_{\varpi_{\varphi}}(\varphi(\theta_{m'p^{\infty}}(\mathcal{D})))$$

where $\varphi(\theta_{m'p^{\infty}}(\mathcal{D})) \in \mathcal{O}_n[\Gamma_{m'p^{\infty}}] \otimes_{\varphi} \mathcal{O}_{\varphi} = \mathcal{O}_{\varphi, n}$.

Remark 15.1. For the $m = p^{\infty}$ case, the parameter $t_{\mathcal{D}}$ is directly affected by analytic Iwasawa μ - and λ -invariants. Indeed, the parameter should come from the λ -invariants only due to vanishing of μ -invariants à la Vatsal.

For the $m = p^r$ case, the parameter $t_{\mathcal{D}}$ can be regarded the mixture of finite analytic Iwasawa μ - and λ -invariants. See [Pol05, Lemma 4.5] for the explicit formula and [Pol05, §4] for the context.

Let $t^* \leq n$ be a non-negative integer. Let $\mathcal{D}_{n+t^*} := (\Delta, g_{n+t^*})$ be an $(n+t^*)$ -admissible form. Then $\mathcal{D}_n := \mathcal{D}_{n+t^*} \pmod{\varpi^n} = (\Delta, g_{n+t^*} \pmod{\varpi^n})$ is also an n -admissible form. Suppose that $t_{\mathcal{D}_n} \leq t^*$.

Remark 15.2 (on the induction parameter). We proceed the induction on $t_{\mathcal{D}_n}$.

If $t_{\mathcal{D}_n}$ is zero or $\text{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi}$ is trivial, then we have nothing to prove. Thus we assume that $t_{\mathcal{D}_n} > 0$ and $\text{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi}$ is non-trivial. Consider the $(n+t_{\mathcal{D}_n})$ -admissible form

$$\mathcal{D} = \mathcal{D}_{n+t_{\mathcal{D}_n}} := (\Delta, f_{n+t^*} \pmod{\varpi^{n+t_{\mathcal{D}_n}}}).$$

Let ℓ be an $(n+t_{\mathcal{D}_n})$ -admissible prime with $\ell \nmid \Delta$. Enlarge $\{\ell\}$ to an $(n+t_{\mathcal{D}_n})$ -admissible set \mathfrak{S} such that $(S, \Delta) = 1$ where $S = \prod_{q \in \mathfrak{S}} q$. Recall the cohomology class

$$\kappa_{\mathcal{D}}(\ell)_{m'p^{\infty}} \in \text{Sel}_{\Delta \ell}(K(m'p^{\infty}), T_{f,n+t_{\mathcal{D}_n}}) \subset \text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n+t_{\mathcal{D}_n}})$$

attached to \mathcal{D} and ℓ as in §11.

By the freeness theorem (Theorem 14.8), the module $\mathcal{M}_n := \text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ is free of rank $\#\mathfrak{S}$ over $\mathcal{O}_{\varphi}/\varphi(\varpi^n)$ for all n . Define the **specialized cohomology class**

$$\kappa_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}} \in \mathcal{M}_{n+t_{\mathcal{D}_n}}$$

by the image of $\kappa_{\mathcal{D}}(\ell)_{m'p^{\infty}}$ under the specialization

$$\text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n+t_{\mathcal{D}_n}}) \rightarrow \mathcal{M}_{n+t_{\mathcal{D}_n}} = \text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n+t_{\mathcal{D}_n}}) \otimes_{\varphi} \mathcal{O}_{\varphi}.$$

Define

$$e_{\mathcal{D}}(\ell) := \text{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}})$$

which may also depend on the choice of \mathfrak{S} .

We have the following (in)equalities:

$$\begin{aligned} e_{\mathcal{D}}(\ell) &\leq \text{ord}_{\varpi_{\varphi}}(\partial_{\ell} \kappa_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}}) && \partial_{\ell} \text{ is a homomorphism.} \\ &= \text{ord}_{\varpi_{\varphi}}(\varphi(\theta_{m'p^{\infty}}(\mathcal{D}))) && \text{The first explicit reciprocity law (Theorem 12.3)} \\ &\leq \text{ord}_{\varpi_{\varphi}}(\varphi(\theta_{m'p^{\infty}}(\mathcal{D}_n))) && \text{Reduction modulo } \varpi^n \text{ is a homomorphism.} \\ &= t_{\mathcal{D}_n} \end{aligned}$$

In fact, the last inequality is equality since $\mathcal{M}_{n+t_{\mathcal{D}_n}}/\varpi_{\varphi} \mathcal{M}_{n+t_{\mathcal{D}_n}} \simeq \mathcal{M}_n/\varpi_{\varphi} \mathcal{M}_n$. See also the proof of [CH15, Lemma 6.16].

15.3. Reduction of the specialized cohomology classes. Choose an element $\tilde{\kappa}_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}} \in \mathcal{M}_{n+t_{\mathcal{D}_n}}$ which satisfies

$$\varpi^{e_{\mathcal{D}}(\ell)} \cdot \tilde{\kappa}_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}} = \kappa_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}}.$$

Then $\tilde{\kappa}_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}}$ is well-defined modulo the $\varpi^{e_{\mathcal{D}}(\ell)}$ -torsion subgroup of $\mathcal{M}_{n+t_{\mathcal{D}_n}}$. Since the torsion subgroup is contained in the kernel of ϖ^n reduction map $\mathcal{M}_{n+t_{\mathcal{D}_n}} \rightarrow \mathcal{M}_n$ with $e_{\mathcal{D}}(\ell) \leq t_{\mathcal{D}_n}$, the **reduced cohomology class** $\kappa'_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}}$ defined by the image under the reduction map

$$\begin{aligned} \text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n+t_{\mathcal{D}_n}}) \otimes_{\varphi} \mathcal{O}_{\varphi} &\longrightarrow \text{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi} \\ \tilde{\kappa}_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}} &\longmapsto \kappa'_{\mathcal{D}, \varphi}(\ell)_{m'p^{\infty}} \end{aligned}$$

is well-defined in \mathcal{M}_n .

Proposition 15.3 (Local properties of the reduced cohomology classes; [CH15, Lemma 6.16]). *The reduced cohomology class $\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty} \in \text{Sel}_\Delta^S(K(m'p^\infty), T_{f,n}) \otimes_\varphi \mathcal{O}_\varphi$ satisfies the following properties:*

- (1) $\text{ord}_{\varpi_\varphi}(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty}) = 0$;
- (2) $\text{ord}_{\varpi_\varphi}(\partial_\ell(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty})) = t_{\mathcal{D}_n} - e_{\mathcal{D}}(\ell)$;
- (3) $\partial_q(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty}) = 0$ for all $q \nmid \Delta\ell$;
- (4) $\text{res}_q(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty}) \in H_{\text{ord}}^1(K(m'p^\infty)_q, T_{f,n})$ for all $q \mid \Delta\ell$.

Remark 15.4. (1) Property (1) in the proposition comes from the isomorphism

$$\mathcal{M}_{n+t_{\mathcal{D}_n}}/\varpi_\varphi \mathcal{M}_{n+t_{\mathcal{D}_n}} \simeq \mathcal{M}_n/\varpi_\varphi \mathcal{M}_n$$

obtained by Corollary 14.15.

- (2) Property (2) is a direct consequence of the first explicit reciprocity law (Theorem 12.3).
- (3) Property (3) and (4) come from Proposition 11.2 and the freeness of $H_{\text{ord}}^1(K(m'p^\infty)_q, T_{f,n+t_{\mathcal{D}_n}})$ by Proposition 5.9.(5).

Remark 15.5. It seems difficult to obtain such reduced cohomology classes without the specialization via φ since the reduction process without the specialization may harm the freeness of compact Selmer groups.

15.4. The first step of induction. We define a natural homomorphism η_ℓ by

$$\begin{aligned} \eta_\ell : H_{\text{sing}}^1(K(m'p^\infty)_\ell, T_{f,n}) &\rightarrow \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \\ \kappa &\mapsto (\eta_\ell(\kappa) : s \mapsto \langle \kappa, v_\ell(s) \rangle_\ell) \end{aligned}$$

and its specialization η_ℓ^φ via φ

$$\eta_\ell^\varphi : H_{\text{sing}}^1(K(m'p^\infty)_\ell, T_{f,n}) \otimes_\varphi \mathcal{O}_\varphi \rightarrow \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi.$$

We need a lemma, which uses the global property of the reduced cohomology classes. It is a variant of [CH15, Lemma 6.17] over $K(m'p^\infty)$.

Lemma 15.6. *The map $\eta_\ell^\varphi(\partial_\ell(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty})) : \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \rightarrow E/\mathcal{O}$ is the zero map.*

Proof. Let $s \in \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})[\ker \varphi]$. By the local properties of the reduced class $\kappa'_{\mathcal{D}}(\ell)_{m'p^\infty}$ (Lemma 15.3.(3), (4)), we see that

$$\langle \partial_q(\kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^\infty}), v_q(s) \rangle_q = 0$$

for all $q \neq \ell$. Then by global reciprocity (Proposition 5.13), the lemma follows. \square

We generalize [BD05, Proposition 4.7] to the general cyclic case.

Proposition 15.7 (The first step of induction). *If $t_{\mathcal{D}_n} = 0$, then $\text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi$ is trivial.*

Note that $t_{\mathcal{D}_n} = 0$ is not equivalent to $\theta_{m'p^\infty}(\mathcal{D}_n) \in \mathcal{O}_n[\Gamma_{m'p^\infty}]^\times$ anymore since $\mathcal{O}_n[\Gamma_{m'p^\infty}]$ is not a local ring.

Proof. We will see $\eta_\ell^\varphi = 0$ but $\eta_\ell^\varphi \pmod{\mathfrak{m}_\varphi} \neq 0$ under the assumption $\text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \neq \{0\}$ in the following commutative diagram

$$\begin{array}{ccc} H_{\text{sing}}^1(K(m'p^\infty)_\ell, T_{f,n}) & \xrightarrow{\eta_\ell} & \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \\ \downarrow \otimes_\varphi \mathcal{O}_\varphi & & \downarrow \otimes_\varphi \mathcal{O}_\varphi \\ H_{\text{sing}}^1(K(m'p^\infty)_\ell, T_{f,n}) \otimes_\varphi \mathcal{O}_\varphi & \xrightarrow[\text{ (=0) }]{\eta_\ell^\varphi} & \text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \\ \downarrow (\text{mod } \mathfrak{m}_\varphi) & & \downarrow (\text{mod } \mathfrak{m}_\varphi) \\ (H_{\text{sing}}^1(K(m'p^\infty)_\ell, T_{f,n}) \otimes_\varphi \mathcal{O}_\varphi) / \mathfrak{m}_\varphi & \xrightarrow[\text{ (} \neq 0 \Rightarrow \Leftarrow \text{)}]{\eta_\ell^\varphi \pmod{\mathfrak{m}_\varphi}} & (\text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) / \mathfrak{m}_\varphi. \end{array}$$

Since the following diagram

$$\begin{array}{ccc} \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) & \xrightarrow{\otimes_{\varphi} \mathcal{O}_{\varphi}} & \mathrm{Sel}_{\Delta}^S(K(m'p^{\infty}), T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi} \\ \downarrow \partial_{\ell} & & \downarrow \partial_{\ell} \\ \mathrm{H}_{\mathrm{sing}}^1(K(m'p^{\infty})_{\ell}, T_{f,n}) & \xrightarrow{\otimes_{\varphi} \mathcal{O}_{\varphi}} & \mathrm{H}_{\mathrm{sing}}^1(K(m'p^{\infty})_{\ell}, T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi} \end{array}$$

commutes, we have

$$\partial_{\ell}(\kappa_{\mathcal{D},\varphi}(\ell)_{m'p^{\infty}}) = \varphi(\partial_{\ell}(\kappa_{\mathcal{D}}(\ell)_{m'p^{\infty}}))$$

under the identification $\mathrm{H}_{\mathrm{sing}}^1(K(m'p^{\infty})_{\ell}, T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi} \simeq \mathcal{O}_n[\Gamma_{m'p^{\infty}}] \otimes_{\varphi} \mathcal{O}_{\varphi} \simeq \mathcal{O}_{\varphi,n}$.

If $\varphi(\theta_{m'p^{\infty}}(\mathcal{D}_n))$ is a unit in $\mathcal{O}_{\varphi,n}$, then the class $\partial_{\ell}(\kappa_{\mathcal{D},\varphi}(\ell)_{m'p^{\infty}})$ generates $\mathrm{H}_{\mathrm{sing}}^1(K(m'p^{\infty})_{\ell}, T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi} \simeq \mathcal{O}_{\varphi,n}$ for any n -admissible prime ℓ via the first explicit reciprocity law (Theorem 12.3). Thus, η_{ℓ}^{φ} is trivial for any n -admissible prime due to Lemma 15.6 with $\kappa_{\mathcal{D},\varphi}(\ell)_{m'p^{\infty}} = \kappa'_{\mathcal{D},\varphi}(\ell)_{m'p^{\infty}}$ under $t_{\mathcal{D}_n} = 0$. Note that \mathcal{D}_n and \mathcal{D} are congruent modulo ϖ^n .

Suppose that

$$\mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi}$$

is non-trivial. Then Nakayama's lemma with \mathcal{O}_{φ} implies that

$$(\mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi}) / \mathfrak{m}_{\varphi}$$

is also non-trivial.

Let $\gamma_{p^r} = (\gamma_m)^{m'}$. Since $\varphi(\gamma_{p^r}) - 1$ is divisible by ϖ_{φ} , we have

$$\begin{aligned} (\mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi}) / \mathfrak{m}_{\varphi} &= (\mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n}))^{\vee} / \mathfrak{m}_{p^{\infty},n} \otimes_{\varphi} \mathcal{O}_{\varphi} / \mathfrak{m}_{\varphi} \\ &= (\mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})[\mathfrak{m}_{p^{\infty},n}])^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi} / \mathfrak{m}_{\varphi} \\ &= (\mathrm{Sel}_{\Delta}(K(m'), A_{f,1}))^{\vee} \otimes_{\varphi} \mathcal{O}_{\varphi} / \mathfrak{m}_{\varphi} \end{aligned} \quad \text{Corollary 14.5}$$

where $\mathfrak{m}_{p^{\infty},n} = (\gamma_{p^{\infty}} - 1, \varpi) \subset \mathcal{O}_n[\Gamma_{m'p^{\infty}}]$.

Let $s \in \mathrm{Sel}_{\Delta}(K(m'), A_{f,1}) \subseteq \mathrm{H}^1(K(m'), A_{f,1})$ be a non-trivial element. By Theorem 13.1, we can choose an n -admissible prime ℓ not dividing $m'p\Delta N^+$ such that $\partial_{\ell}(s) = 0$ and $v_{\ell}(s) \neq 0 \in \mathrm{H}_{\mathrm{fin}}^1(K(m')_{\ell}, A_{f,1})$. For such s and ℓ , $\kappa \mapsto \langle \kappa, v_{\ell}(s) \rangle_{\ell}$ cannot be identically zero on $\kappa \in \mathrm{H}_{\mathrm{sing}}^1(K(m'), T_{f,1})$ since the local Tate pairing (in $\mathrm{H}_{\mathrm{fin}}^1(K(m')_{\ell}, A_{f,1})$) at any prime ℓ is perfect. This implies that $\eta_{\ell}^{\varphi} \pmod{\mathfrak{m}_{\varphi}}$ cannot be trivial for such an ℓ . A contradiction follows considering the commutative diagram given first. \square

15.5. Appealing to the second explicit reciprocity law. Due to Proposition 15.7, we may assume $t_{\mathcal{D}_n} > 0$. Let Π be the set of $(n + t^*)$ -admissible primes ℓ satisfying

- (1) $\ell \nmid m'p\Delta N^+$,
- (2) the integer $e_{\mathcal{D}}(\ell) = \mathrm{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D},\varphi}(\ell)_{m'p^{\infty}})$ is minimal among ℓ satisfying the condition (1).

Then Π is non-empty by Theorem 13.1. Let $e_{\min} := e_{\mathcal{D}}(\ell)$ for any $\ell \in \Pi$.

Lemma 15.8 ([CH15, Lemma 6.19]). *The strict inequality $e_{\min} < t_{\mathcal{D}_n}$ holds.*

Proof. Suppose that $e_{\min} = t_{\mathcal{D}_n}$. Then $e_{\mathcal{D}}(\ell) = t_{\mathcal{D}_n}$ for all $(n + t^*)$ -admissible primes ℓ due to inequality $e_{\mathcal{D}}(\ell) \leq t_{\mathcal{D}_n}$. By Corollary 14.5, we have $\mathrm{H}^1(K, A_{f,1}) \simeq \mathrm{H}^1(K(m'p^{\infty}), A_{f,n})[\mathfrak{m}_{m'p^{\infty},n}]$. Then there exists a non-zero element $s \in \mathrm{Sel}_{\Delta}(K(m'p^{\infty}), A_{f,n})[\mathfrak{m}_{m'p^{\infty},n}] \subseteq \mathrm{H}^1(K, A_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$. By Theorem 13.1 with $m' = 1$, there exists an $(n + t^*)$ -admissible prime ℓ such that $v_{\ell}(s)$ is non-zero in $\mathrm{H}_{\mathrm{fin}}^1(K_{\ell}, A_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$. Also, by Lemma 15.3.(2), the image of $\partial_{\ell}(\kappa'_{\mathcal{D},\varphi}(\ell))$ in $\mathrm{H}_{\mathrm{sing}}^1(K_{\ell}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ is non-zero. By Lemma 15.6, the image of $\partial_{\ell}(\kappa'_{\mathcal{D},\varphi}(\ell))$ in $\mathrm{H}_{\mathrm{sing}}^1(K_{\ell}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ and $v_{\ell}(s)$ are orthogonal to each other with respect to the local Tate pairing. Since the local Tate pairing $\mathrm{H}_{\mathrm{fin}}^1(K_{\ell}, A_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi} \times \mathrm{H}_{\mathrm{sing}}^1(K_{\ell}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi} \rightarrow \mathcal{O}_{\varphi} / \varpi \mathcal{O}_{\varphi}$ is perfect, and $\mathrm{H}_{\mathrm{fin}}^1(K_{\ell}, A_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ and $\mathrm{H}_{\mathrm{sing}}^1(K_{\ell}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ are one-dimensional over $\mathcal{O}_{\varphi} / \varpi \mathcal{O}_{\varphi}$, we have a contradiction. \square

Let $\ell_1 \in \Pi$ and \mathfrak{S} be an $(n+t^*)$ -admissible set with $S = \prod_{q \in \mathfrak{S}} q$. Let $\kappa_{m'} \in H^1(K(m'), T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$ be the image of $\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}$ under the map

$$\text{Sel}_{\Delta}^S(K(m'p^\infty), T_{f,n}) \otimes_{\varphi} (\mathcal{O}_{\varphi}/\varpi_{\varphi} \mathcal{O}_{\varphi}) \xrightarrow{\simeq} \left(\text{Sel}_{\Delta}^S(K(m'p^\infty), T_{f,n})/\mathfrak{m}_{p^\infty,n} \right) \otimes_{\varphi} \mathcal{O}_{\varphi} \xrightarrow{\text{cores}} H^1(K(m'), T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$$

$$\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty} \longmapsto \kappa_{m'}.$$

The last inclusion comes from Corollary 14.15, which comes from Theorem 14.8.

Therefore, $\kappa_{m'}$ is a non-zero element in $H^1(K(m'), T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$. By Theorem 13.1, there exists an $(n+t^*)$ -admissible prime ℓ_2 not dividing $m'p\Delta N^+$ such that

- $\partial_{\ell_2}(\kappa_{m'}) = 0$, and
- $v_{\ell_2}(\kappa_{m'}) \neq 0$ in $H^1(K(m')_{\ell_2}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi}$.

Then we have $v_{\ell_2}(\kappa_{m'}) \neq 0$. The following commutative diagram

$$\begin{array}{ccc} \text{Sel}_{\Delta}^S(K(m'p^\infty), T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi}/\varpi_{\varphi} \mathcal{O}_{\varphi} & \xrightarrow{\text{cores}} & H^1(K(m'), T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi} \\ \downarrow v_{\ell_2} & & \downarrow v_{\ell_2} \\ H_{\text{fin}}^1(K(m'p^\infty)_{\ell_2}, T_{f,n}) \otimes_{\varphi} \mathcal{O}_{\varphi}/\varpi_{\varphi} \mathcal{O}_{\varphi} & \xrightarrow{\simeq} & H_{\text{fin}}^1(K(m')_{\ell_2}, T_{f,1}) \otimes_{\varphi} \mathcal{O}_{\varphi} \end{array}$$

$$\begin{array}{ccc} \kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty} & \xrightarrow{\quad} & \kappa_{m'} = \text{cores}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}) \\ \downarrow & & \downarrow \\ v_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}) & \xrightarrow{\quad} & v_{\ell_2}(\kappa_{m'}) \neq 0 \end{array}$$

shows that $\text{ord}_{\varpi_{\varphi}}(v_{\ell_2}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) = \text{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})$. In other words, the homomorphism v_{ℓ_2} does not increase the valuation considering Lemma 15.3.(1). Also, the minimality of $e_{\min} = e_{\mathcal{D}}(\ell_1) = \text{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})$ imply that

$$\begin{aligned} \text{ord}_{\varpi_{\varphi}}(v_{\ell_2}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) &= \text{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}) && \text{the property of } v_{\ell_2} \\ (5) \quad &\leq \text{ord}_{\varpi_{\varphi}}(\kappa_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty}) && \text{the minimality of } e_{\min} \\ &\leq \text{ord}_{\varpi_{\varphi}}(v_{\ell_1}(\kappa_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})) && v_{\ell_1} \text{ is a homomorphism.} \end{aligned}$$

By the second explicit reciprocity law (Theorem 12.5), there exists an $(n+t^*)$ -admissible form $\mathcal{D}_{n+t^*}^{\ell_1\ell_2} = (\Delta\ell_1\ell_2, g_{n+t^*})$ such that

$$v_{\ell_2}(\kappa_{\mathcal{D}}(\ell_1)_{m'p^\infty}) = v_{\ell_1}(\kappa_{\mathcal{D}}(\ell_2)_{m'p^\infty}) = \theta_{m'p^\infty}(\mathcal{D}_{n+t^*}^{\ell_1\ell_2})$$

where $\mathcal{D}_{n+t^*}^{\ell_1\ell_2} = (\Delta\ell_1\ell_2, g_{n+t^*} \pmod{\varpi^{n+\mathcal{D}_n}})$. This directly implies that $\text{ord}_{\varpi_{\varphi}}(v_{\ell_1}(\kappa_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})) = \text{ord}_{\varpi_{\varphi}}(v_{\ell_2}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}))$. Therefore, we have $\text{ord}_{\varpi_{\varphi}}(v_{\ell_1}(\kappa_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})) = \text{ord}_{\varpi_{\varphi}}(v_{\ell_2}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}))$. Then all the inequalities (5) become equalities and we have

$$\text{ord}_{\varpi_{\varphi}}(v_{\ell_2}(\kappa_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) = e_{\mathcal{D}}(\ell_1) = e_{\mathcal{D}}(\ell_2) = e_{\min}$$

and $\ell_2 \in \Pi$.

Let $\mathcal{D}_n^{\ell_1\ell_2} := (\Delta\ell_1\ell_2, g_{n+t^*} \pmod{\varpi^n})$. Then we have

$$t_{\mathcal{D}_n^{\ell_1\ell_2}} = \text{ord}_{\varpi_{\varphi}}(\theta_{m'p^\infty}(\mathcal{D}_n^{\ell_1\ell_2})) = e_{\min} < t_{\mathcal{D}_n} \leq t^*$$

due to Lemma 15.8.

Thus, we can apply the induction hypothesis to $\mathcal{D}_{n+t^*}^{\ell_1\ell_2}$ reminding Remark 15.2. In other words, we may assume to have the following statement.

Assumption 15.9 (Induction hypothesis).

$$\varphi(L_p(K(m'p^\infty), \mathcal{D}_{n+t^*}^{\ell_1\ell_2})) \in \text{Fitt}_{\mathcal{O}_{\varphi,n+t^*}}(\text{Sel}_{\Delta\ell_1\ell_2}(K(m'p^\infty), A_{f,n+t^*})^\vee \otimes_{\varphi} \mathcal{O}_{\varphi}).$$

15.6. Reduction for induction. Let $S_{[\ell_1\ell_2]} \subseteq \text{Sel}_{\Delta}(K(m'p^\infty), A_{f,n})$ be the subgroup consisting of classes which are locally trivial at the primes dividing ℓ_1 and ℓ_2 . Then we have two exact sequences of $\mathcal{O}[[\Gamma_{m'p^\infty}]]$ -modules

$$\begin{array}{ccccccc} H_{\text{sing}}^1(K(m'p^\infty)_{\ell_1}, T_{f,n}) \oplus H_{\text{sing}}^1(K(m'p^\infty)_{\ell_2}, T_{f,n}) & \xrightarrow{\eta_s} & \text{Sel}_{\Delta}(K(m'p^\infty), A_{f,n})^\vee & \longrightarrow & S_{[\ell_1\ell_2]}^\vee & \longrightarrow & 0 \\ H_{\text{fin}}^1(K(m'p^\infty)_{\ell_1}, T_{f,n}) \oplus H_{\text{fin}}^1(K(m'p^\infty)_{\ell_2}, T_{f,n}) & \xrightarrow{\eta_f} & \text{Sel}_{\Delta\ell_1\ell_2}(K(m'p^\infty), A_{f,n})^\vee & \longrightarrow & S_{[\ell_1\ell_2]}^\vee & \longrightarrow & 0 \end{array}$$

where η_s and η_f are induced by the local paring $\langle -, - \rangle_{\ell_1} \oplus \langle -, - \rangle_{\ell_2}$. Fix isomorphisms

$$\begin{aligned} H_{\text{sing}}^1(K(m'p^\infty)_{\ell_1}, T_{f,n}) \oplus H_{\text{sing}}^1(K(m'p^\infty)_{\ell_2}, T_{f,n}) &\simeq \mathcal{O}_n[[\Gamma_{m'p^\infty}]] \oplus \mathcal{O}_n[[\Gamma_{m'p^\infty}]] \\ H_{\text{fin}}^1(K(m'p^\infty)_{\ell_1}, T_{f,n}) \oplus H_{\text{fin}}^1(K(m'p^\infty)_{\ell_2}, T_{f,n}) &\simeq \mathcal{O}_n[[\Gamma_{m'p^\infty}]] \oplus \mathcal{O}_n[[\Gamma_{m'p^\infty}]]. \end{aligned}$$

Tensoring with \mathcal{O}_φ via φ , two sequences become

$$\begin{aligned} \mathcal{O}_{\varphi,n} \oplus \mathcal{O}_{\varphi,n} &\xrightarrow{\eta_s^\varphi} \mathrm{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0 \\ \mathcal{O}_{\varphi,n} \oplus \mathcal{O}_{\varphi,n} &\xrightarrow{\eta_f^\varphi} \mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0 \end{aligned}$$

where η_s^φ and η_f^φ are induced from η_s and η_f , respectively, as in §15.4.

By Lemma 15.6, we know that η_s^φ factors through

$$\mathcal{O}_{\varphi,n} / (\partial_{\ell_1}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) \oplus \mathcal{O}_{\varphi,n} / (\partial_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})).$$

In other words, the first sequence becomes

$$\mathcal{O}_{\varphi,n} / (\partial_{\ell_1}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) \oplus \mathcal{O}_{\varphi,n} / (\partial_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})) \xrightarrow{\eta_s^\varphi} \mathrm{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0.$$

By Lemma 15.3.(2), we have

$$t_{\mathcal{D}_n} - t_{\mathcal{D}_n^{\ell_1\ell_2}} = \mathrm{ord}_{\varpi_\varphi}(\partial_{\ell_1}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) = \mathrm{ord}_{\varpi_\varphi}(\partial_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})).$$

The first sequence can be restated as

$$\left(\mathcal{O}_{\varphi,n} / \varpi_\varphi^{t_{\mathcal{D}_n} - t_{\mathcal{D}_n^{\ell_1\ell_2}}} \right)^2 \xrightarrow{\eta_s^\varphi} \mathrm{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0.$$

In order to deal with the second sequence, we need the following lemma.

Lemma 15.10. *The kernel of η_f^φ contains $(0, v_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}))$ and $(v_{\ell_1}(\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty}), 0)$.*

Proof. The proof is similar to the one of Lemma 15.6. See [CH15, Lemma 6.20] for detail. \square

The second sequence becomes

$$\mathcal{O}_{\varphi,n} / (v_{\ell_1}(\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty})) \oplus \mathcal{O}_{\varphi,n} / (v_{\ell_2}(\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty})) \xrightarrow{\eta_f^\varphi} \mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0.$$

Note that we have

$$\mathrm{ord}_{\varpi_\varphi}(v_{\ell_2}\kappa'_{\mathcal{D},\varphi}(\ell_1)_{m'p^\infty}) = \mathrm{ord}_{\varpi_\varphi}(v_{\ell_1}\kappa'_{\mathcal{D},\varphi}(\ell_2)_{m'p^\infty}) = t_{\mathcal{D}_n^{\ell_1\ell_2}} - e_{\min} = 0.$$

This shows that

$$\mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \simeq S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi.$$

Plugging this isomorphism into the first sequence, we have

$$\left(\mathcal{O}_\varphi / \varpi_\varphi^{t_{\mathcal{D}_n} - t_{\mathcal{D}_n^{\ell_1\ell_2}}} \right)^2 \xrightarrow{\eta_s^\varphi} \mathrm{Sel}_\Delta(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow \mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \longrightarrow 0.$$

Let \mathcal{S} be the exact kernel of the map $\mathrm{Sel}_\Delta(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi \rightarrow \mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi$ in the sequence. Then we have

$$\begin{aligned} \varphi(L_p(K(m), \mathcal{D}_n)) &= \varpi_\varphi^{2t_{\mathcal{D}_n}} \\ &= \varpi_\varphi^{2(t_{\mathcal{D}_n} - t_{\mathcal{D}_n^{\ell_1\ell_2}})} \cdot \varpi_\varphi^{2t_{\mathcal{D}_n^{\ell_1\ell_2}}} \\ &\in \mathrm{Fitt}_{\mathcal{O}_{\varphi,n}}(\mathcal{S}) \cdot \mathrm{Fitt}_{\mathcal{O}_{\varphi,n}}(\mathrm{Sel}_{\Delta\ell_1\ell_2}(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) \quad \text{Assumption 15.9} \\ &= \mathrm{Fitt}_{\mathcal{O}_{\varphi,n}}(\mathcal{S}) \cdot \mathrm{Fitt}_{\mathcal{O}_{\varphi,n}}(S_{[\ell_1\ell_2]}^\vee \otimes_\varphi \mathcal{O}_\varphi) \\ &\subseteq \mathrm{Fitt}_{\mathcal{O}_{\varphi,n}}(\mathrm{Sel}_\Delta(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi). \quad \text{Lemma 4.3} \end{aligned}$$

Remark 15.11. More precisely, the induction hypothesis (Assumption 15.9) is used modulo ϖ^n with the Selmer group argument given in §16.

To sum up, we obtain the following theorem with the first step of induction (Proposition 15.7).

Theorem 15.12. *Assume the following conditions:*

- (1) $(\bar{\rho}, \Delta)$ satisfies Condition CR;
- (2) f satisfies Condition PO;
- (3) f is N^+ -minimal;
- (4) $\bar{\rho}$ has big image.

For any n -admissible form $\mathcal{D}_n = (\Delta, g_n)$ and any homomorphism $\varphi : \mathcal{O}[\Gamma_m] \rightarrow \mathcal{O}_\varphi$ where \mathcal{O}_φ is a discrete valuation ring of characteristic zero, we have

$$\varphi(L_p(K(m'p^\infty), \mathcal{D}_n)) \in \text{Fitt}_{\mathcal{O}_{\varphi,n}}(\text{Sel}_\Delta(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi).$$

16. COMPLETION OF PROOF: LIFTING TO CHARACTERISTIC ZERO

In this section, we complete the proof of the main theorem (Theorem 1.18) with help of Proposition 6.2. Of course, all the conditions in the main theorems are assumed in this section. We deal with the case $\mathcal{D}_n^f = (\Delta, g_n) = (N^-, f_{\alpha,n})$.

First, we observe that the equality of ideals

$$\text{Fitt}_{\mathcal{O}_{\varphi,n}}(\text{Sel}_{N^-}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) = \text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) \pmod{\varpi_\varphi^n}$$

holds in $\mathcal{O}_{\varphi,n}$ due to Lemma 4.5.(2).

Consider the following diagram

$$\begin{array}{ccccc} (\varphi(L_p(K(m'p^\infty), f))) & \xrightarrow{\quad \subseteq \quad} & \text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) & \xrightarrow{\quad \subseteq \quad} & \mathcal{O}_\varphi \\ \text{Proposition 12.1} \downarrow (\text{mod } \varpi_\varphi^n) & & \downarrow (\text{mod } \varpi_\varphi^n) & & \downarrow (\text{mod } \varpi_\varphi^n) \\ (\varphi(L_p(K(m'p^\infty), \mathcal{D}_n^f))) & \xrightarrow[\text{Theorem 15.12}]{\quad \subseteq \quad} & \text{Fitt}_{\mathcal{O}_{\varphi,n}}(\text{Sel}_{N^-}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi) & \xrightarrow{\quad \subseteq \quad} & \mathcal{O}_{\varphi,n}. \end{array}$$

Since \mathcal{O}_φ is a discrete valuation ring, all the ideals of \mathcal{O}_φ are filtered. Thus, the inclusion in characteristic zero follows.

Remark 16.1. This lifting argument is a cost for using n -admissible forms. If we work with the p -isolated condition and the rigid pairs, then we can bypass this part since every n -admissible form can be uniquely lifted to a form of characteristic zero. See [BD05, Proposition 3.12 and Theorem 9.3] for detail.

Thus, we have

$$\varphi(L_p(K(m'p^\infty), f)) \in \text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m'p^\infty), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi)$$

for all $n \geq 1$.

By Lemma 4.4, we know

$$\text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m), A_f)^\vee \otimes_\varphi \mathcal{O}_\varphi) = \bigcap_{n=1}^{\infty} \text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m), A_{f,n})^\vee \otimes_\varphi \mathcal{O}_\varphi).$$

Thus, we have inclusion $\varphi(L_p(K(m), f)) \in \text{Fitt}_{\mathcal{O}_\varphi}(\text{Sel}_{N^-}(K(m), A_f)^\vee \otimes_\varphi \mathcal{O}_\varphi)$. Applying Proposition 6.2 to this inclusion, Theorem 1.18 follows.

17. A MORE GLOBAL APPLICATION WITH ANTICYCLOTOMIC PARTIALLY ADELIC L -FUNCTIONS

In this section, we investigate a more global aspect of the Mazur-Tate type conjecture by gluing the main theorem (Theorem 1.18) varying p . The exposition of this section is rather brief.

For the notational convenience, we focus on the case of elliptic curves over \mathbb{Q} .

Let E be an elliptic curve over \mathbb{Q} of conductor N . Let K be an imaginary quadratic field with the same assumption with N as before. Let m be an integer with $(m, N) = 1$. Let $M = \prod_i p_i^{r_i}$ be an integer where each $p_i > 3$ is a prime such that

- (1) each p_i is good ordinary for E ;
- (2) each residual representation $\bar{\rho}_{E,p_i}$ is surjective;
- (3) condition PO holds for $(E, K(m'_i)/K, p_i)$ for each p_i where m'_i is the prime-to- p_i part of m ;
- (4) $N = N(\bar{\rho}_{E,p_i})$ for each p_i .

Then, by Theorem 1.18, we know

$$L_p(K(m), E) \pmod{p^r} \in \text{Fitt}_{(\mathbb{Z}/p^r\mathbb{Z})[\Gamma_m]}(\text{Sel}(K(m), E[p^r])^\vee)$$

for any $p^r \mid M$.

Let f be the newform corresponding to E .

Assumption 17.1. Suppose that we are able to regard f as a “partially global” modular form by making it a $\mathbb{Z}[\frac{1}{6}]$ -valued function

$$f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \rightarrow \mathbb{Z}[\frac{1}{6}]$$

with normalization $f \pmod{p_i} \neq 0$ for each prime factor p_i of M .

Remark 17.2. Note that we need to invert 2 and 3 due to Remark 3.1. Maybe we need to invert more primes or to extend \mathbb{Z} , but we do not go further on this issue. There may not exist such a normalization (at least naturally) if there exists certain Tamagawa defect, i.e. Condition (4) for M breaks.

Applying the same construction of the Bertolini-Darmon element as in §3.2 to this “global” f , we can define an analogous element $L_M(K(m), f) \in \mathbb{Z}[\frac{1}{6}][\Gamma_m]$. Then it is not difficult to see

$$L_M(K(m), f) \pmod{p_i^{r_i}} = L_{p_i}(K(m), f) \pmod{p_i^{r_i}}$$

in $\mathbb{Z}/p_i^{r_i}\mathbb{Z}[\Gamma_m]$ for $p_i^{r_i} \mid M$ under Assumption 17.1.

By Chinese remainder theorem with $E[M]$ and $E(K(m))/M \cdot E(K(m))$, we have

$$\text{Sel}(K(m), E[M])[p_i^{r_i}] \simeq \text{Sel}(K(m), E[p_i^{r_i}])$$

where $M = \prod_i p_i^{r_i}$ as we defined.

With Lemma 14.4 and Condition (4) for M , we also have

$$\text{Sel}(K(m), E[M])[p^r] \simeq \text{Sel}(K(m), E[p^r])$$

for any $p^r \mid M$.

Theorem 17.3. Under Assumption 17.1, we have

$$L_M(K(m), E) \in \text{Fitt}_{\mathbb{Z}/M\mathbb{Z}[\Gamma_m]}(\text{Sel}(K(m), E[M])^\vee)$$

in $\mathbb{Z}/M\mathbb{Z}[\Gamma_m]$ where $(-)^\vee = \text{Hom}(-, \mathbb{Q}/\mathbb{Z})$.

Proof. Let $M = \prod_i p_i^{r_i}$. By Theorem 1.18, we have

$$L_M(K(m), E) \pmod{p_i^{r_i}} \in \text{Fitt}_{\mathbb{Z}/p_i^{r_i}\mathbb{Z}[\Gamma_m]}(\text{Sel}(K(m), E[p_i^{r_i}])^\vee)$$

for each prime divisor p_i of M . By Lemma 4.5.(2), we have

$$\text{Fitt}_{\mathbb{Z}/M\mathbb{Z}[\Gamma_m]}(\text{Sel}(K(m), E[M])^\vee) \pmod{p_i^{r_i}} = \text{Fitt}_{\mathbb{Z}/p_i^{r_i}\mathbb{Z}[\Gamma_m]}(\text{Sel}(K(m), E[p_i^{r_i}])^\vee).$$

By Chinese remainder theorem, we have

$$\mathbb{Z}/M\mathbb{Z}[\Gamma_m] \simeq \left(\prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z} \right) [\Gamma_m].$$

Thus, we obtain the conclusion. □

In the same manner as in Corollary 1.23, we obtain

Corollary 17.4. Suppose that Assumption 17.1 holds. Let $\chi : \Gamma_m \rightarrow \mathbb{Z}_\chi^\times$ be a character and assume that $\chi(L_M(K(m), f)) \neq 0$. Then

$$\sum_{p_i} (\text{ord}_{p_i}(\#|E(K(m))_\chi|) + \text{ord}_{p_i}(\#\text{III}(K(m), E)[M]_\chi)) \leq \sum_{p_i} \text{ord}_{p_i}(\chi(L_M(K(m), f)))$$

where $\text{III}(K(m), E)[M]_\chi$ is the χ -isotypic quotient of the M -torsion subgroup of the Shafarevich-Tate group of E over $K(m)$.

Remark 17.5. Corollary 17.4 gives a partial affirmative answer to the question raised in [BD05, Remark 1 to Corollary 4, Introduction] on the size of $\text{III}(K(m), E_f)[M]$ for not necessarily prime power $M = p^k$.

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